Supplementary Material for
"If You're NOT So Smart, Why Are You Rich?
Robust Market Selection with General Recursive Preferences"

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Overview

This supplement collects material moved out of the main article to meet the target submission length. Section S.1 gives proofs of intermediate lemmas and facts whose statements remain in the main appendix. Section S.2 discusses Assumption 6. Section S.3 proves Theorem 3. Section S.4 constructs ε-perfected economies. Section S.5 contains the DAUA/IAUA robustness results. Section S.6 gives the equilibrium-existence result and proof. Sections S.7 and S.8 deal with the Epstein–Zin case.

S.1 Technical Proofs Delegated from the Main Appendix

This section contains proofs of intermediate lemmas and facts whose statements remain in the main appendix.

Proof of Lemma 2

Proof. By definition, we have

$$\frac{L^{ij}(\sigma_t)}{\hat{L}^{ij}(\sigma_t)} = \frac{\prod_{\tau=0}^{t-1} B^{ji}(\sigma_{\tau+1}) \times D^{ji}(\sigma_\tau)}{\prod_{\tau=0}^{t-1} B^{ji}(\sigma_{\tau+1}) \times D^{ji}(\sigma_{\tau+1})} = \frac{D^{ji}(\sigma_0)}{D^{ji}(\sigma_t)}.$$

Note that

$$D^{ji}(\sigma_t) = \frac{F_2^j\left(c(\sigma_t), \mathcal{R}_t^j\left(V_{t+1}^j\right)\right)}{F_2^i\left(c(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right)} \in (0, +\infty)$$

by Assumption 7. So, there exists $0 < \underline{d} < \overline{d} < +\infty$ that bound D^{ij} from below and above, which proves the lemma. □

Proof of Lemma 3

Proof. First, note that V_t^i is continuous in product topology, so for all $\varepsilon > 0$, there exists some $K \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\max_{0\le k\le K}\max_{\omega\in\Sigma(\sigma_t)} c_{t+k}^i(\omega) < \epsilon \Longrightarrow |V_t^i - \underline{V}^i| < \varepsilon. \tag{S.1.1}$$

Second, we can show that fixing $\epsilon > 0$ and $K \in \mathbb{N}$, there exists some $L, L_0 \in \mathbb{R}_+$ such that

$$L_t^{ij} > L \Rightarrow \min\left\{L_t^{ij}, \ldots, L_{t+K}^{ij}\right\} > L_0 \Rightarrow \max\left\{c_t^i, \ldots, c_{t+K}^i\right\} < \epsilon. \tag{S.1.2}$$

The first implication comes from the fact that L_{t+k}^{ij} is L_t^{ij} multiplied by $B^{ji} \times D^{ji}$ in the following k periods, where D^{ji} is bounded by Assumption 7 and B^{ji} is also bounded.[1] The second implication comes from the fact that for all $\tau \in \mathbb{T}$,

$$L_\tau^{ij} > L_0 \Rightarrow F_1^i\left(c_\tau^i, \mathcal{R}_\tau^i\right) > L_0 \times F_1^j\left(c_\tau^j, \mathcal{R}_\tau^j\right),$$

and

$$F_1^j\left(c_\tau^j, \mathcal{R}_\tau^j\right) > \min\left\{F_1^j(\overline{e}, V) : V \in \mathcal{V}^j\right\} \equiv \underline{F}_1^j > 0,$$

where $\mathcal{V}^j = \left[\underline{V}^j, \max_s \overline{V}_s^j\right]$ stands for the set of possible utility values for consumer j. Note that $\mathcal{R}_\tau^i \in \mathcal{V}^i$ has a compact range, and F^i satisfies the Inada condition, so we can choose L_0 sufficiently large such that

$$F_1^i\left(c_\tau^i, \mathcal{R}_\tau^i\right) > L_0 \times \underline{F}_1^j \Rightarrow c_\tau^i < \epsilon,$$

which proves (S.1.2). Lemma 2 then implies that there exists $\overline{L}$ such that

$$\min\left\{\hat{L}_t^{ij}, \ldots, \hat{L}_{t+K}^{ij}\right\} > \overline{L} \Longrightarrow \min\left\{L_t^{ij}, \ldots, L_{t+K}^{ij}\right\} > L_0, \tag{S.1.3}$$

so Lemma 3 is proved after combining (S.1.1), (S.1.2) and (S.1.3). □

Proof of Lemma 4

Proof. For $L < \infty$ and $t \in \mathbb{T}$, define

$$A_t(L) \equiv \left\{\hat{L}_\tau^{-i} > L \text{ for all } \tau \ge t\right\}. \tag{S.1.4}$$

Define the local discount-rate and belief ratios relevant for the dynamics of $\hat{L}^{ji}$ by

$$D_i^{ij}(s) \equiv \frac{F_2^i\left(e(s), \mathcal{R}^i\left(\overline{V}^i\right)\right)}{F_2^j\left(0, \mathcal{R}^j\left(\underline{V}^j\right)\right)} \quad \text{and} \quad B_i^{ij}(s) \equiv \frac{\nabla_s \mathcal{R}^i\left(\overline{V}^i\right)}{\nabla_s \mathcal{R}^j\left(\underline{V}^j\right)}. \tag{S.1.5}$$

[1] Since $\mathcal{R}$ is continuously differentiable and strictly increasing, $\nabla\mathcal{R} > 0$ is also continuous. Also, continuation utility lies on a compact set. Therefore, B^{ji} is a continuous function over a compact set and hence is bounded.

Their reciprocals are denoted by D_i^{ji} and B_i^{ji}. Since $j \succ_i i$, letting $\Delta \equiv \mathcal{S}_i^j - \mathcal{S}_i^i > 0$, we have

$$\mathbb{E} \log \left[B_i^{ij}(s) D_i^{ij}(s) \right] = \mathcal{S}_i^i - \mathcal{S}_i^j = -\Delta.$$

Choose $\varepsilon > 0$ such that $\log(1+\varepsilon) < \Delta$. By Lemma 3, market clearing, the finiteness of S, and continuity on the compact ranges of continuation utilities, there exists $\overline{L} < \infty$, independent of t, such that, on $A_t(\overline{L})$,

$$B^{ij}(\sigma_\tau, s_{\tau+1}) D^{ij}(\sigma_\tau, s_{\tau+1}) \leq (1+\varepsilon) B_i^{ij}(s_{\tau+1}) D_i^{ij}(s_{\tau+1})$$

for every $\tau \geq t$. This is a uniform relative bound because the local multipliers are strictly positive and S is finite. Hence, on $A_t(\overline{L})$,

$$\begin{aligned} \log \hat{L}_{\tau+1}^{ji} &= \log \hat{L}_\tau^{ji} + \log \left[B^{ij}(\sigma_\tau, s_{\tau+1}) D^{ij}(\sigma_\tau, s_{\tau+1}) \right] \\ &\leq \log \hat{L}_\tau^{ji} + \log \left[B_i^{ij}(s_{\tau+1}) D_i^{ij}(s_{\tau+1}) \right] + \log(1+\varepsilon). \end{aligned}$$

Iterating this inequality and applying the strong law of large numbers gives

$$\begin{aligned} \limsup_{T \to +\infty} \frac{1}{T-t} \log \hat{L}_T^{ji} &\leq \mathbb{E} \log \left[B_i^{ij}(s) D_i^{ij}(s) \right] + \log(1+\varepsilon) \\ &= \mathcal{S}_i^i - \mathcal{S}_i^j + \log(1+\varepsilon) = -\Delta + \log(1+\varepsilon) < 0 \quad \mathbb{P}\text{-a.s. on } A_t(\overline{L}). \end{aligned} \tag{S.1.6}$$

Thus, outside a null event, $A_t(\overline{L})$ would imply $\hat{L}_T^{ji} \to 0$. But on $A_t(\overline{L})$,

$$\hat{L}_T^{ji} \geq \hat{L}_T^{-i} > \overline{L} \quad \text{for every } T \geq t,$$

a contradiction. Therefore, $\mathbb{P}(A_t(\overline{L})) = 0$ for every t, which proves (13). □

Proof of Fact 2

Proof. Let $X(s) = B_i^{ji}(s) \times D_i^{ji}(s)$, which is a bounded random variable. Then,

$$\lim_{\rho \to 0+} \mathbb{E} \left(\frac{X^\rho - 1}{\rho} \right) = \mathbb{E} \left(\lim_{\rho \to 0+} \frac{X^\rho - 1}{\rho} \right) = \mathbb{E} \left(\lim_{\rho \to 0+} X^\rho \times \log X \right) = \mathbb{E} \log X < 0,$$

where the first equality uses the dominated convergence theorem and the second one uses L'Hôpital rule. Since $\mathbb{E}\left(\frac{X^\rho - 1}{\rho}\right)$ is continuous in ρ, there exists $\rho > 0$ such that $\mathbb{E}\left(\frac{X^\rho - 1}{\rho}\right) < 0$, that is $\mathbb{E}X^\rho < 1$. □

Proof of Fact 3

Proof. Let ρ satisfy Fact 2. When $\hat{L}_0^{i-} > \underline{L}$, the process is constant, and hence trivially a supermartingale. Suppose that $\hat{L}_0^{i-} \leq \underline{L}$. From Lemma 3, when $\underline{L}$ is very small and when $\hat{L}_t^{i-} \leq \underline{L}$, both $B^{ji}(\sigma_t, s)$ and $D^{ji}(\sigma_t, s)$ are very close to $B_i^{ji}(s)$ and $D_i^{ji}(s)$. So for all

$\varepsilon > 0$, there exists $\underline{L} > 0$ such that when $t < T$,

$$\mathbb{E}\left[m_{t+1}^{ij}|\mathcal{F}_t\right] \leq m_t^{ij} \times (1+\varepsilon)\,\mathbb{E}\left[\left(B_i^{ji}(s) \times D_i^{ji}(s)\right)^{\rho}\right].$$

By Fact 2 we can choose $\varepsilon > 0$ to be sufficiently small so that $\mathbb{E}\left[m_{t+1}^{ij}|\mathcal{F}_t\right] \leq m_t^{ij}$ when $t < T$. Thus, $\left\{m_t^{ij}\right\}$ is a supermartingale. □

Proof of Lemma 6

Proof. For all $t \in \mathbb{T}$ and given that $\hat{L}_t^{i-} < \varepsilon$, then we define a stopping time $T = \inf\left\{\tau > t : \hat{L}_\tau^{i-} \geq \varepsilon'\right\}$, where $\varepsilon' > \varepsilon > 0$. For all i and j, define a stopped process $m_{t+k}^{ij} = \left(\hat{L}_{\min\{t+k,T\}}^{ij}\right)^{\rho}$. Fact 3 shows that there exists $\rho, \varepsilon' > 0$ such that $\left\{m_{t+k}^{ij}\right\}$ is a supermartingale, and hence converges to some limit m_∞^{ij}. Doob's optional stopping theorem implies that

$$\varepsilon^{\rho} \geq \left(\hat{L}_t^{ij}\right)^{\rho} \geq \mathbb{E}\left(m_\infty^{ij}|\mathcal{F}_t\right) \geq (\varepsilon')^{\rho} \times \mathbb{P}_{\{T<\infty\}}\left(\hat{L}_T^{ij} \geq \varepsilon'|\mathcal{F}_t\right).$$

Therefore, we have

$$\mathbb{P}_{\{T<\infty\}}\left(\max_{j\in D_i} \hat{L}_T^{ij} \geq \varepsilon' \middle| \mathcal{F}_t\right) \leq \sum_{j\in D_i} \mathbb{P}_{\{T<\infty\}}\left(\hat{L}_T^{ij} \geq \varepsilon' \middle| \mathcal{F}_t\right) \leq I\left(\frac{\varepsilon}{\varepsilon'}\right)^{\rho}. \tag{S.1.7}$$

If $U_i \neq \emptyset$, then Lemma 4 implies that $\mathbb{P}(T = \infty) = 0$, that is,

$$\mathbb{P}\left(\hat{L}_T^{i-} \geq \varepsilon' \middle| \mathcal{F}_t\right) = 1,$$

where $\hat{L}_\tau^{i-} = \frac{1}{\hat{L}_\tau^{-i}}$, so (S.1.7) implies that

$$\begin{aligned} &\mathbb{P}\left(\max_{j\in D_i} \hat{L}_T^{ij} \geq \varepsilon'|\mathcal{F}_t\right) \leq I\left(\frac{\varepsilon}{\varepsilon'}\right)^{\rho} \\ \Longrightarrow &\mathbb{P}\left(\max_{j\in U_i} \hat{L}_T^{ij} \geq \varepsilon' \middle| \mathcal{F}_t\right) \geq 1 - I\left(\frac{\varepsilon}{\varepsilon'}\right)^{\rho}, \end{aligned} \tag{S.1.8}$$

Here, we choose ε sufficiently small relative to ε', so the probability on the R.H.S. of (S.1.8) is strictly positive, which implies the lemma. □

Proof of Lemma 7

Proof. From Lemma 6, there exists $p > 0$ such that

$$\mathbb{P}\left(\cup_{j\in U_i}\left\{\hat{L}_\tau^{ij} \geq \varepsilon' \text{ for some } \tau > t\right\}|L_t^{i-} < \varepsilon\right) \geq p. \tag{S.1.9}$$

Define $T = \inf\left\{\tau > t : \hat{L}_\tau^{i-} \geq \varepsilon'\right\}$. So we have: (i) $T < \infty$, and (ii) there exists some $j \in U_i$ such that $\hat{L}_T^{ij} \geq \varepsilon'$. Each such j must satisfy

$$\hat{L}_T^{j-} = \max_{k\neq j} \hat{L}_T^{jk} = \hat{L}_T^{ji} \times \max_{k\neq j} \hat{L}_T^{ik}$$

$$\leq \frac{1}{\varepsilon'} \times \varepsilon' \max\left(D^{ki}B^{ki}\right) \leq \frac{1}{\varepsilon'} \times \varepsilon' M \text{ for some } M < \infty. \tag{S.1.10}$$

where the boundedness of D and B comes from the proof of Lemma 2. (S.1.9) and (S.1.10) imply that

$$\mathbb{P}\left(\cup_{j\in U_i}\left\{\hat{L}_\tau^{j-} \leq \varepsilon' M \text{ for some } \tau \geq t\right\} | L_t^{i-} < \varepsilon\right) \geq p.$$

Assumption 6 implies that conditional on $\hat{L}_t^{j-} \leq \varepsilon' M$, there exists $K < \infty$ signals after which we have $\hat{L}_{\tau+K}^{j-} < \varepsilon$. So, there exists some $q > 0$ such that probability of $\hat{L}_\tau^{j-} < \varepsilon$ for some $\tau > t$ is uniformly bounded by q. □

Proof of Lemma 8

Proof. Let $\hat{L} \equiv \left\{\hat{L}^{ij}, \text{ for all } i, j \in I\right\}$ and we define $B_\varepsilon^i = \left\{\hat{L} : \hat{L}^{i-} < \varepsilon\right\}$. From Assumption 6, we know that for all $t \in \mathbb{T}$, the probability that there exists some $j_1 \in I$ such that $\hat{L}_{t_1} \in B_\varepsilon^{j_1}$ for some $t_1 \geq t$ is uniformly greater than 0.[2] Then, when $\hat{L}_{t_1} \in B_\varepsilon^{j_1}$, and if $U_{j_1} \neq \emptyset$, Lemma 7 implies that with a strictly positive probability that exists some $t_2 > t_1$ such that $\hat{L}_{t_2} \in B_\varepsilon^{j_2}$ for some $j_2 \in U_{j_1}$. Applying the arguments iteratively, we know that with a strictly positive probability, $\hat{L}_t$ follows the trajectory $B_\varepsilon^{j_1} \to B_\varepsilon^{j_2} \to B_\varepsilon^{j_3} \to \ldots$, where $j_{m+1} \in U_{j_m}$ until it enters some $B_\varepsilon^{j_*}$ with $U_{j_*} = \emptyset$. Once $\hat{L}_t \in B_\varepsilon^{j_*}$, Lemma 5 implies that $\hat{L}_t$ is trapped inside $B_\varepsilon^{j_*}$ with a strictly positive probability, in which case consumer j^* dominates the market. Note that in each step, the probability is uniformly bounded away from 0. Therefore, for all $t \in \mathbb{T}$ and $\mathcal{F}_t$, there exists some $p > 0$ and $i \in I^*$ such that

$$\mathbb{P}\left(\cup_{i\in I^*}\left\{c_t^j \to 0 \text{ for all } j \neq i\right\} | \mathcal{F}_t\right) > p,$$

i.e., the probability that some consumer in I^* eventually dominates the market is uniformly greater than 0. Levy's 0-1 law then implies that with probability 1, some consumer in I^* will eventually dominate. Besides, from Lemma 5 and Assumption 6, we know that each consumer in I^* dominates with a strictly positive probability. □

Proof of Lemma 9

Proof. Euler's homogeneous function theorem implies that for all $k \in \mathbb{R}_{++}$, we have

$$k = \mathcal{R}(k \cdot \mathbf{1}) = k \times \frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial k} = k \times \sum_{s\in S} \frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial V_s}$$

$$\Rightarrow \sum_{s\in S} \frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial V_s} = 1.$$

[2] We simply choose $j_1 \in \arg\min_{j\in I} F_1^j\left(c_t^j, \mathcal{R}_t^j\right)$. By definition we have $L_t^{j_1-} = \max_i L_t^{ji} \leq 1 < \infty$. Assumption 6 implies that the probability that $L_{t_1}^{j_1-} < \varepsilon$ for some $t_1 > t$ is uniformly bounded from below.

Assumption 2 says that $\frac{\partial \mathcal{R}(k\cdot\mathbf{1})}{\partial V_s} > 0$, so $\nabla\mathcal{R}(k\cdot\mathbf{1})$ is a probability distribution. □

Proof of Lemma 10

Proof. Notice that

$$\begin{aligned}\mathbb{E}\log\left[\frac{\nabla_s\mathcal{R}^j\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right] &= \mathbb{E}\log\left[\frac{\left(1-\varepsilon_j\right)\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)+\varepsilon_j\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right] \\ &\geq \left(1-\varepsilon_j\right)\times\mathbb{E}\log\left(\frac{\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right)+\varepsilon_j\mathbb{E}\log\left(\frac{\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right) \\ &= \varepsilon_j\mathbb{E}\log\left(\frac{\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right) > 0. \end{aligned} \tag{S.1.11}$$

□

Proof of Lemma 11

Proof. Construct the following utility function

$$u_j(x) \equiv u_i(x) + \beta\left[\mathcal{R}^i\left(\overline{V}^i\right) - \mathcal{R}^j\left(\overline{V}^i\right)\right].$$

Notice that $C_j \equiv \mathcal{R}^i\left(\overline{V}^i\right) - \mathcal{R}^j\left(\overline{V}^i\right) \to 0$ as $\varepsilon_j \to 0$. So, when ε_j is sufficiently small, we have $||u_j - u_i|| < \delta$. It is easy to verify that $u_j \in \mathscr{U}$ when ε_j is sufficiently small (because u_j is equal to some constant plus $u_i \in \mathscr{U}$, and when ε_j is sufficiently small, C_j is also small, so we can guarantee that $u_j(0) > 0$ as well). Notice that for all $s \in S$, we have

$$\begin{aligned}\overline{V}^i_s = u_i(e(s)) + \beta\times\mathcal{R}^i\left(\overline{V}^i\right) &= u_j(e(s)) - \beta\left[\mathcal{R}^i\left(\overline{V}^i\right) - \mathcal{R}^j\left(\overline{V}^i\right)\right] + \beta\times\mathcal{R}^i\left(\overline{V}^i\right) \\ &= u_j(e(s)) + \beta\mathcal{R}^j\left(\overline{V}^i\right),\end{aligned}$$

so $\overline{V}^i$ solves $V = F\left(e, \mathcal{R}^j(V)\right)$. By definition, we also have $\overline{V}^j = F\left(e, \mathcal{R}^j\left(\overline{V}^j\right)\right)$. Under our assumptions, $F\left(c, \mathcal{R}^j(V)\right)$ has a unique fixed point, so $\overline{V}^j = \overline{V}^i$. □

S.2 Discussion on Assumption 6

Below is a condition on exogenous variables that is sufficient for Assumption 6 to hold.

Proposition S.2.1. *Let $0 < d < \infty$ be a uniform upper bound on D^{ji}. Suppose there exists*

some $M > 0$ with $Md < 1$ such that for all $i \in I$, $t \in \mathbb{T}$, and $\sigma_t \in \Sigma_t$,

$$\sup_{V,W \in \mathcal{V}} \left[\max_{j \neq i} \frac{\nabla_s \mathcal{R}_t^j(V)}{\nabla_s \mathcal{R}_t^i(W)} \right] \leq M \quad \text{for some } s \in S$$

having positive conditional probability, where $\mathcal{V}$ denotes a compact set containing all possible continuation-utility vectors. Then Assumption 6 holds.

Proof. By definition,

$$\begin{aligned} \max_{j \neq i} L_{t+1}^{ij} &\leq \max_{j \neq i} L_t^{ij} \max_{j \neq i} \frac{\nabla_s \mathcal{R}_t^j(V_{t+1}^j)}{\nabla_s \mathcal{R}_t^i(V_{t+1}^i)} \max_{j \neq i} D^{ji}(\sigma_t) \\ &\leq \max_{j \neq i} L_t^{ij} Md \quad \text{when } s_{t+1} = s. \end{aligned}$$

Since $Md < 1$, choose $K \in \mathbb{N}$ such that $\overline{L}(Md)^K < \underline{L}$. After K successive realizations of the corresponding states, we have $\max_{j \neq i} L_{t+K}^{ij} < \underline{L}$, so Assumption 6 holds. □

Proposition S.2.1 says that one sufficient condition for Assumption 6 to hold is that there exists some state $s \in S$ at which every consumer's effective belief is sufficiently different from other consumers. To get a more concrete idea, we employ Proposition S.2.1 to provide conditions for smooth ambiguity preferences below.

Example S.2.1 (Smooth ambiguity preferences)**.** Suppose that consumers have smooth ambiguity preferences with i.i.d. beliefs,

$$V_t(c) = u(c_t) + \beta \times \phi^{-1}\left(\mathbb{E}_\mu \phi\left[\mathbb{E}_\pi\left(V_{t+1}(c)\right)\right]\right).$$

Denote by $m = \min_i \frac{\phi_i'(\overline{V})}{\phi_i'(\underline{V})}$, $M = \max_i \frac{\phi_i'(\underline{V})}{\phi_i'(\overline{V})}$ and $d = \frac{\min_i \beta^i}{\max_i \beta^i}$, where $\underline{V}$ and $\overline{V}$ denote the lowest and the highest possible utility levels. If for all i, we have

$$\max_{j \neq i} \frac{\pi^j(s)}{\pi^i(s)} < \frac{m}{M} \times d \quad \text{for some } s \in S, \tag{S.2.1}$$

where $\pi^i(s) \equiv \sum_\pi \mu^i(\pi) \times \pi(s)$, then Assumption 6 is satisfied.

S.3 Proof of Theorem 3

Denote by E the event that all consumers in $I \backslash G$ vanish. Under the assumptions of Theorem 3, we have the following lemmas.

Lemma S.3.1. *On E, $\mathbb{P}$–almost surely there exists some $i \in G$ such that $\hat{L}_t^{i-}$ does not converge.*

Proof. Suppose not. Then it happens with a strictly positive probability that $\hat{L}_t^{i-}$ converges for all $i \in G$. We first notice that by Assumption 6, the limit $\hat{L}_\infty^{i-}$ can only be 0 or ∞ except

for null events. We further note that on E, all consumers in $I\backslash G$ vanish, and since $U_i \neq \emptyset$ for all $i \in G$, no consumer in G dominates the market by Proposition 3, so $\hat{L}^{i-}_\infty$ cannot be zero. Therefore, $\hat{L}^{i-}_\infty$ must be ∞ for all $i \in G$. But then all consumers in G vanish as well, which is not possible by definition. □

Lemma S.3.2. *Suppose that $\hat{L}^{i-}_t$ does not converge, then* $\liminf \hat{L}^{i-}_t = 0$ *except for $\mathbb{P}$-null events.*

Proof. Denote by G the event that $\hat{L}^{i-}_t$ does not converge, so we can rewrite it as

$$G = \cup_{m,n\in\mathbb{Z}_{++}} \left\{ \hat{L}^{i-}_t \text{ crosses } \left[\frac{1}{m}, n\right] \text{ infinitely many times} \right\} \equiv \cup_{m,n\in\mathbb{Z}_{++}} G_{mn}.$$

Further denote by F the event that $\liminf \hat{L}^{i-}_t > 0$, which can be rewritten as

$$F = \cup_{k\in\mathbb{Z}_{++}} \left\{ \liminf \hat{L}^{i-}_t \geq 1/k \right\} \equiv \cup_{k\in\mathbb{Z}_{++}} F_k.$$

Therefore, $G \cap F = \cup_{k,m,n\in\mathbb{Z}_{++}} G_{mn} \cap F_k$ denotes the event that $\hat{L}^{i-}_t$ does not converge and $\liminf \hat{L}^{i-}_t > 0$. If $k \leq m$, then by definition $G_{mn} \cap F_k$ is a null event. If $k > m$, then on G_{mn}, $\hat{L}^{i-}_t$ falls below $1/m$ infinitely often. By Assumption 6, whenever $\hat{L}^{i-}_t \leq 1/m$, then it will fall below $1/k$ with a strictly positive probability in the future. So, on G_{mn}, we have

$$\sum_t \mathbb{P}\left(\hat{L}^{i-}_\tau < 1/k \text{ for some } \tau > t | \mathcal{F}_t\right) = +\infty.$$

By Levy's extension of Borel-Cantelli Lemmas, we almost surely have $\left\{\hat{L}^{i-}_t < 1/k \text{ i.o.}\right\}$ on G_{mn}, which implies $\mathbb{P}\left(G_{mn} \cap F_k\right) = 0$. The arguments hold for all $k, m, n \in \mathbb{Z}_{++}$, so we further have $\mathbb{P}\left(G \cap F\right) = 0$, which implies that when $\hat{L}^{i-}_t$ does not converge, we have $\liminf \hat{L}^{i-}_t = 0$ except for null events. □

Proof of Theorem 3

Proof. Combining Lemmas S.3.1 and S.3.2, we know that on E, there almost surely exists $i \in G$ such that $\hat{L}_t \in B^i_\varepsilon = \left\{\hat{L} \in \mathbb{R}^{I\times I}_{++} : \hat{L}^{i-} < \varepsilon\right\}$ happens infinitely often. Recall that Lemma 6 says that

$$\mathbb{P}\left(\max_{j\in U_i} \hat{L}^{ij}_{T^i_t} \geq \varepsilon' \middle| \hat{L}^{i-}_t < \varepsilon \right) \geq p,$$

where $T^i_t = \inf\left\{\tau > t : \hat{L}^{i-}_\tau \geq \varepsilon'\right\}$. Recall that T^i_t is $\mathbb{P}$-almost surely finite. Therefore on E, we almost surely have

$$\sum_t \mathbb{P}\left(\max_{j\in U_i} \hat{L}^{ij}_{T^i_t} \geq \varepsilon' \middle| \mathcal{F}_t \right) = +\infty$$

which, by Levy's extension of Borel-Cantelli Lemma, implies that $\max_{j\in U_i} \hat{L}^{ij}_t \geq \varepsilon'$ occurs infinitely often on E almost surely. By definition, all consumers in $I\backslash G$ vanish on E, so

we have $\max_{j\in U_i\cap G}\hat{L}_t^{ij}\geq\varepsilon'$ occurs infinitely often on E.[3] For any $t\in\mathbb{T}$, suppose that $\hat{L}_t^{ij}\geq\varepsilon'$ for some $j\in G$, then $\hat{L}_t$ enters B_ε^j with a positive probability. Following the iterated arguments as in Lemma 8, $\hat{L}_t$ follows a trajectory $B_\varepsilon^{j_1}\to B_\varepsilon^{j_2}\to B_\varepsilon^{j_3}\to\ldots$, where $j_{m+1}\in U_{j_m}\cap G$ for each m and finally enters B_ε^g for some $g\in G^*$ with a positive probability, and the probability is bounded away from 0 for all t. Therefore, on E, we have

$$\sum_t\mathbb{P}\left(\cup_{g\in G^*}\left\{\hat{L}_\tau\in B_\varepsilon^g\text{ for some }\tau>t\right\}|\mathcal{F}_t\right)=+\infty,$$

which implies that $\hat{L}_t\in B_\varepsilon^g$ for some $g\in G^*$ infinitely often almost surely. Lemma 6 says that conditional on $\hat{L}_t\in B_\varepsilon^g$, the probability that $\max_{j\in U_g}\hat{L}_{T_t^g}^{gj}\geq\varepsilon'$. Therefore, on E, we have

$$\sum_t\mathbb{P}\left(\cup_{g\in G^*}\left\{\max_{j\in U_g}\hat{L}_{T_t^g}^{gj}\geq\varepsilon'\right\}|\mathcal{F}_t\right)=+\infty,\tag{S.3.1}$$

which implies that $\hat{L}_{T_t^g}^{gj}\geq\varepsilon'$ for some $g\in G^*$and $j\in U_g$ infinitely often almost surely. Notice that by assumption, $U_g\neq\emptyset$ and $U_g\subset I/G$, so (S.3.1) implies $\hat{L}_{T_t^g}^{gj}\geq\varepsilon'$ infinitely often for some $g\in G^*$and $j\in I/G$, which further implies, once again, that some consumer $j\in I/G$ does not vanish on E almost surely, so E must be a null event. □

S.4 Construction of ε-Perfected Economy

Let tuple $\mathcal{E}_S=(e,\pi_0,F,\mathcal{R})$ denote the original economy, where $e=\{e_i\}_{i\in I}$, $F=\left\{F^i\right\}_{i\in I}$, and $\mathcal{R}=\left\{\mathcal{R}^i\right\}_{i\in I}$ denote the profiles of endowment, time aggregators, and certainty equivalent aggregators, and the subscript S emphasizes that the state space is S.[4] Consider another economy with an extended state space $\hat{S}\supset S$ and tuple $\mathcal{E}_{\hat{S}}=\left(\hat{e},\hat{\pi}_0,\hat{F},\hat{\mathcal{R}}\right)$. Next we want to describe the distance between $\mathcal{E}_{\hat{S}}$ and $\mathcal{E}_S$. Note that $\mathcal{E}_S$ is defined on a smaller domain, but we can describe all elements $\mathcal{E}_S$ in the same domain as $\mathcal{E}_{\hat{S}}$ by natural extension. The distance $||\mathcal{E}_S-\mathcal{E}_{\hat{S}}||$ is then defined as the maximum distance between the coordinates of $\mathcal{E}_{\hat{S}}$ and $\mathcal{E}_S$ on the extended domain. Formally,

Definition S.4.1. For economy $\mathcal{E}_S=(e,\pi_0,F,\mathcal{R})$ and $\mathcal{E}_{\hat{S}}=\left(\hat{e},\hat{\pi}_0,\hat{F},\hat{\mathcal{R}}\right)$ with $\hat{S}\supset S$. We

[3] If there were some $j\in U_i\cap G^c$ such that $\hat{L}_{T_t^i}^{ij}\geq\varepsilon'$ occurred infinitely often, then j's consumption would be bounded away from zero infinitely often, contradicting the definition of E. Indeed, at $T=T_t^i$, we have $\hat{L}_{T-1}^{i-}<\varepsilon'$ and $\hat{L}_T^{ij}\geq\varepsilon'$. Comparability and the uniform boundedness of the one-period multipliers therefore imply $F_{1,T}^j\leq C\min_k F_{1,T}^k$ for some $C<\infty$. Market clearing and compactness uniformly bound $\min_k F_{1,T}^k$ from above; hence the Inada condition gives $c_T^j\geq\delta>0$, where δ is independent of T.

[4] Throughout this section, the discount rate for all consumers is equal to β (Assumption 10) and the set of consumers is always I, so we remove them from the definition of an economy for brevity.

define

$$||\mathcal{E}_S - \mathcal{E}_{\hat{S}}|| \equiv \max\{d_e, d_\pi, d_F, d_R\},$$

where

$$d_e \equiv \max_{i\in I, s\in\hat{S}} |e_i(s) - \hat{e}_i(s)|, \text{ where } e_i(s) \equiv 0 \text{ for } s \in \hat{S}\backslash S$$

$$d_\pi \equiv \max_{s\in\hat{S}} |\pi_0(s) - \hat{\pi}_0(s)|, \text{ where } \pi_0(s) \equiv 0 \text{ for } s \in \hat{S}\backslash S$$

$$d_F \equiv \max_{c,y\in C\times Y} |F(c,y) - \hat{F}(c,y)|,$$

$$d_R \equiv \max_{i\in I}\left\{\max_{V\in\mathcal{V}}\left|\mathcal{R}^i(V|_S) - \hat{\mathcal{R}}^i(V)\right|, \max_{V\in\mathcal{V}}\left|\mathbb{E}_{\pi_0}\log\nabla_s\mathcal{R}^i(V|_S) - \mathbb{E}_{\hat{\pi}_0}\log\nabla_s\hat{\mathcal{R}}^i(V)\right|\right\},$$

where $C \equiv [0,\overline{e}]$ and $Y \equiv [0,\overline{v}]$, where $\overline{e}$ is the maximum total endowment in these two economies and $\overline{v}$ denotes the maximum utility, and $\mathcal{V}$ is a compact subset of $[0,\overline{\boldsymbol{v}}] \subset \mathbb{R}^{\hat{S}}$ containing all relevant continuation-utility vectors and on which the displayed gradients are strictly positive, where $\overline{\boldsymbol{v}} = \overline{v}\times 1 \in \mathbb{R}^{\hat{S}}$, and $V|_S \equiv (V_s)_{s\in S}$ is the restriction of V on S.

Definition S.4.2. We say that $\mathcal{E}_{\hat{S}}$ is an ***ε-perfection*** of $\mathcal{E}_S$ if we have: (i) $||\mathcal{E}_S - \mathcal{E}_{\hat{S}}|| < \varepsilon$, and (ii) $\mathcal{E}_{\hat{S}}$ satisfies Assumptions 1 to 6.

The second component of d_R controls expected log marginal continuation values directly, since the logarithm is not continuous near zero; together with fixed-point continuity, it implies convergence of every local survival index. The existence of an ε-perfected economy is guaranteed by the lemma below.

Lemma S.4.1. (Existence) *For all $\mathcal{E}_S$ satisfying Assumptions 1 to 5 and all $\varepsilon > 0$, there exists an ε-perfection of $\mathcal{E}_S$.*

Proof. The proof is by construction. Let $\hat{S} = S \cup \{\hat{s}_1, \ldots, \hat{s}_I\}$. For some $\eta > 0$, define

$$\hat{e}_i(s) = \begin{cases} e_i(s) & \text{when } s \in S \\ \eta & \text{when } s \in \hat{S}\backslash S \end{cases}, \text{ and } \hat{\pi}_0(s) = \begin{cases} (1-\eta)\pi_0(s) & \text{when } s \in S \\ \eta/I & \text{when } s \in \hat{S}\backslash S \end{cases}.$$

Thus $\hat{\pi}_0$ is a probability distribution. It is straightforward that $\hat{e}$ and $\hat{\pi}_0$ satisfy all relevant assumptions when η is sufficiently small. We keep F unchanged, so $\hat{F}^i = F^i$. For any vector $V \subset \mathbb{R}^{\hat{S}}_+$, define

$$\hat{\mathcal{R}}^i(V) = \left(1-\eta-\eta^2\right)\mathcal{R}^i(V|_S) + \eta V_{\hat{s}_i} + \sum_{j\neq i}\frac{\eta^2}{I-1}V_{\hat{s}_j}.$$

By assumption, we have: (i) $\mathcal{R}^i$ is continuously differentiable and strictly increasing in V_s for $s \in S$, and (ii) $\mathcal{R}^i$ satisfies $\mathcal{R}^i(k) = k$. Therefore, $\hat{\mathcal{R}}^i$ also satisfies these properties on the extended domain. As $\eta \downarrow 0$, the primitive distances converge to zero. The old-state log

terms converge uniformly; on the new states, probabilities are η/I and gradients are η or $\eta^2/(I-1)$, giving an $O(\eta|\log\eta|)$ contribution to the second component of d_R. Last, we show that the construction satisfies Assumption 6. Recall that consumption dynamics satisfy

$$L_{t+1}^{ij} = L_t^{ij} \times B^{ji}(\sigma_t, s_{t+1}) \times D^{ji}(\sigma_t).$$

For all $i \in I$, when $s_{t+1} = \hat{s}_i$, we have

$$B^{ji}(\sigma_t, s_{t+1}) = \frac{\nabla_{s_{t+1}} \hat{\mathcal{R}}^j(V_{t+1}^j)}{\nabla_{s_{t+1}} \hat{\mathcal{R}}^i(V_{t+1}^i)} = \frac{\eta}{I-1}, \quad \forall j \neq i.$$

Besides, D^{ji} is bounded by some constant $\overline{d}$ from the proof of Lemma 2. Therefore, when η is sufficiently small, finitely many realizations of $\hat{s}_i$ make L_t^{i-} fall below any given threshold. Thus the construction satisfies Assumption 6, and choosing η sufficiently small makes the distance smaller than ε. □

S.5 Robust Survival and Decreasing Absolute Uncertainty Aversion

We first introduce the following definition.

Definition S.5.1. $\mathcal{R}^i$ displays ***decreasing (increasing, constant) absolute uncertainty aversion*** if $k - \mathcal{R}^i(V + k \cdot \mathbf{1})$ is a decreasing (increasing, constant) function of $k \in \mathbb{R}$.

Intuitively, $k - \mathcal{R}^i(V + k \cdot \mathbf{1})$ is a measure of the utility loss due to the presence of uncertainty V. When the loss in utility decreases with k, we say the preferences display decreasing uncertainty aversion. The definition above is equivalent to decreasing absolute risk aversion for the case of expected utility preferences (see Proposition 6.C.3 in Mas-Colell et al. (1995)) and to decreasing ambiguity aversion for the case of ambiguity-averse preferences (see Fabbri et al. (2024)).[5]

Lemma S.5.1. *Suppose Assumption 2 holds and $\mathcal{R}^i$ displays decreasing (increasing, constant) absolute uncertainty aversion. Then, $\sum_s \nabla_s \mathcal{R}^i(V) \geq (\leq, =)\ 1$.*

To gain intuition, recall that in the case of pure risk, decreasing absolute uncertainty aversion implies a precautionary saving motive (see Kimball (1990)) and, therefore, that the consumer's marginal utility of a certain future income is greater in the presence of uncertainty than in its absence as in Leland (1968).[6] In our more general case with

[5] It is easy to see that decreasing uncertainty aversion implies the certainty equivalent is constant superadditive, i.e. $\mathcal{R}(V + k \cdot \mathbf{1}) \geq \mathcal{R}(V) + k$ for all $k \geq 0$ (see Cerreia-Vioglio et al. (2022))

[6] For the case of ambiguity, Berger (2014) and Osaki and Schlesinger (2014) find conditions such that decreasing absolute ambiguity aversion implies a precautionary motive as well.

uncertainty, decreasing uncertainty aversion implies that the marginal utility of a constant continuation value in the presence of uncertainty, $\sum_s \nabla_s \mathcal{R}(V)$, is greater than its marginal utility without uncertainty, $\sum_s \nabla_s \mathcal{R}(k \cdot \mathbf{1})$, which is equal to 1 (see Lemma 9).

Proposition S.5.1. *Under Assumptions 1–5 and 8–10, if $\mathcal{R}^i$ displays increasing or constant absolute uncertainty aversion, then it cannot robustly survive.*

Proof. If $\mathcal{R}^i$ displays increasing or constant absolute uncertainty aversion, then by Lemma S.5.1, we have $\sum_s \nabla_s \mathcal{R}^i\left(\overline{V}^i\right) \leq 1$. Let β denote the common discount factor. Then

$$\begin{aligned}
\mathcal{S}_i^i &= \mathbb{E}\log \nabla_s \mathcal{R}^i\left(\overline{V}^i\right) + \mathbb{E}\log\beta \\
&= \mathbb{E}\log\left(\frac{\nabla_s \mathcal{R}^i\left(\overline{V}^i\right)}{\sum_s \nabla_s \mathcal{R}^i\left(\overline{V}^i\right)}\right) + \mathbb{E}\log\left(\sum_s \nabla_s \mathcal{R}^i\left(\overline{V}^i\right)\right) + \mathbb{E}\log\beta \\
&\leq \mathbb{E}\log\left(\pi_0\left(s\right)\right) + \mathbb{E}\log\left(1\right) + \mathbb{E}\log\beta = \mathcal{S}_i^r,
\end{aligned}$$

so $r \succeq_i i$, equivalently $i \not\succ_i r$. By Theorem 4, $\mathcal{R}^i$ cannot survive robustly. □

Intuitively, Lemma S.5.1 implies that with increasing absolute uncertainty aversion, the sum of effective beliefs is less than 1 in the presence of uncertainty, so consumers become endogenously less patient. Therefore, consumers with such preferences cannot locally dominate a more patient consumer with correct SEU preferences, and hence by Theorem 4, those preferences cannot robustly survive.

S.6 Existence of the Equilibrium

In this section, we provide conditions under which a competitive equilibrium exists. We do not impose continuity in product topology directly; instead, we show that this property can also be proved under our assumptions.

Assumption S.6.1. *$F^i\left(x, y\right)$ is concave and the equation $F^i\left(x, y\right) = y$ has at least one non-negative solution for every $x \geq 0$ and $F^i\left(0, 0\right) > 0$.*[7]

Assumption S.6.2. *$\mathcal{R}_t^i\left(V\right)$ is concave.*

Assumption 3 simply says that endowments are uniformly bounded. Assumptions S.6.1 and S.6.2 impose conditions on F and $\mathcal{R}$ that are used to prove continuity and convexity of

[7]The last part of Assumption S.6.1 rules out the case where the minimum utility is 0 as it requires $F^i\left(0, 0\right) > 0$. This requirement of Assumption S.6.1 is for technical convenience, it could be relaxed in some interesting cases, e.g., with a separable time aggregator. To establish existence of equilibrium, we could allow $F^i\left(0, 0\right) = 0$ if we add the assumption that F is a Blackwell contraction in y.

the preference relation, a key step to show the existence of an equilibrium. Now we are ready to state our result on the existence of a competitive equilibrium in our economy.

Theorem S.6.1. *Suppose Assumptions 1 to 4, S.6.1 and S.6.2 hold, then a competitive equilibrium exists and satisfies* (2).

Next we prove Theorem S.6.1. Our proof is based on the existence theorem by Bewley (1969) that requires preferences to be strongly monotone, weakly convex and continuous in the Mackey topology. We adapt the arguments in Marinacci and Montrucchio (2010) to show that recursive utility functions generated by a time aggregator and a certainty equivalent aggregator satisfying those assumptions are continuous in the product topology.[8] Since the product topology is coarser than the Mackey topology, it follows that preferences are continuous in the Mackey topology as well. We also show that preferences are strongly monotone and weakly convex.

S.6.1 Some Useful Results

The economy has countably many commodities, so we can use the following theorem due to Bewley (1969) to establish the existence of a competitive equilibrium in our economy.[9]

Theorem S.6.2. *Let $E = (C, (C_i, \succeq_i, e_i)_{i \in I})$ be an economy which satisfies the following:*

1) $C = L_\infty(B, \mathcal{B}, \upsilon)$ where $(B, \mathcal{B}, \upsilon)$ is a positive measure space.

2) $\forall i$, $C_i = x_i + K$, where $K = \{c \in C : c \geq 0\}$ is the positive cone of C.

3) $\forall i$, $e_i \gg x_i$,

4) $\forall i$, the pre-order $\succeq_i$ is

(a) Mackey continuous: for each $\underline{c} \in C_i$, the sets $\{c \in C_i : c \succeq_i \underline{c}\}$ and $\{c \in C_i : c \precsim_i \underline{c}\}$ are closed in C_i with respect to the Mackey topology for the dual pair (L_∞, L_1),

(b) weakly convex: for each $\underline{c} \in C_i$, the set $\{c \in C_i : c \succeq_i \underline{c}\}$ is convex.

(c) strongly monotone: $c \in C_i$ and $k > 0$ implies that $c + k \succ_i c$,

(d) complete: if $c_1, c_2 \in C_i$, then either $c_1 \succeq_i c_2$ or $c_2 \succeq_i c_1$.

Then E has a competitive equilibrium.

Next, we introduce the following definitions used by Marinacci and Montrucchio (2010).

[8] The existence and uniqueness of a utility function satisfying recursion (1) follows almost directly from Theorem 1(ii) in Marinacci and Montrucchio (2010), albeit with a slight modification. Marinacci and Montrucchio (2010) restricts attention to consumption plans c that are bounded away from 0 (i.e., $[c]_\infty > 0$ in their notation). Since we are interested in the case where $c_t \to 0$, we cannot restrict to the case in which $\lfloor c \rfloor_\infty > 0$. Instead, we assume $F^i(0,0) > 0$ and show that their result still holds.

[9] The existence theorem in Bewley (1969) focuses on an exchange economy, which can be directly applied here. See Bewley (1972) for an existence theorem that includes production as well.

Definition S.6.1. F is *γ-subhomogeneous* if there exists some $\gamma > 0$ such that

$$F(\alpha^{\gamma} x, \alpha y) \geq \alpha F(x, y)$$

for all $\alpha \in (0, 1]$ and all $x, y \in \mathbb{R}_+$.

Definition S.6.2. F is a *Thompson aggregator* if $F(x, 0) > 0$ for all $x > 0$, and

$$F(x, \alpha y) \geq \alpha F(x, y) + (1 - \alpha) F(x, 0)$$

for all $\alpha \in [0, 1]$ and all $x, y \in \mathbb{R}_+$.

Definition S.6.3. $\mathcal{R}$ is *subhomogeneous* if $\mathcal{R}_t(\alpha V_{t+1}) \geq \alpha \mathcal{R}_t(V_{t+1})$ for all $\alpha \in [0, 1]$, $t \geq 0$ and all adapted processes $V = \{V_t : t \in \mathbb{T}\}$.

An example of a certainty equivalent function that is subhomogeneous can be found in Example 1. The following fact is easy to verify.

Lemma S.6.1. *Under Assumptions 1, S.6.1 and S.6.2, F is a Thompson aggregator and γ-subhomogeneous, and $\mathcal{R}$ is subhomogeneous.*

Proof. (i) Thompson: From Assumption 1, $F \geq 0$ and $F_1, F_2 > 0$, we must have $F(x, 0) > 0$ for all $x > 0$; the second condition comes from concavity of F, i.e., Assumption S.6.1. (ii) γ-subhomogeneous: The concavity and non-negativity of F imply that

$$F(\alpha x, \alpha y) \geq \alpha F(x, y) + (1 - \alpha) F(0, 0) \geq \alpha F(x, y),$$

then we simply let $\gamma = 1$. (iii) Similarly, $\mathcal{R}$ is subhomogeneous because

$$\mathcal{R}_t(\alpha V_{t+1}) \geq \alpha \mathcal{R}_t(V_{t+1}) + (1 - \alpha) \mathcal{R}_t(0) \geq \alpha \mathcal{R}_t(V_{t+1}),$$

where the first inequality comes from concavity, and the second comes from non-negativity. □

S.6.2 Proof of Theorem S.6.1

Let $c \in L_{\infty}^{+}$ be a consumption plan and $T : L^{+} \to L^{+}$ be the operator given by

$$T(V) = F(c, \mathcal{R}(V)), \tag{S.6.1}$$

where L^{+} denotes the set of non-negative adapted processes. That is, for all $t \geqslant 0$, $T_t(V) = F(c_t, \mathcal{R}_t(V_{t+1}))$. For a fixed consumption sequence $c \in L_{\infty}^{+}$, let T^c be the operator defined in (S.6.1). The next lemma shows the existence of a unique $V^c \in L^{+}$ that solves $V = T^c(V)$ and that the mapping $c \mapsto V^c$ is continuous in the product topology.

Lemma S.6.2. *Under Assumptions 1 to 4, S.6.1 and S.6.2, there exists a unique $V^c \in L^{+}$ that solves equation $V = T^c(V)$ and the mapping $c \mapsto V_t^c$ is pointwise continuous in the product topology for all $t \geq 0$.*

Proof. Existence and uniqueness follows from Theorem 1 in Marinacci and Montrucchio (2010). We only need to show that property (ii) in their Theorem 10 holds since properties (i) and (iii) are not affected by our assumption that $F(0,0) > 0$. By Assumption S.6.1 there exists y^* such that $F(\|c\|_\infty, y^*) = y^*$. Let $X \in [0, Y^*]$ where $Y_t^* = y^*$ for all $t \geq 0$. Let $\alpha = \frac{F(0,0)}{y^*}$ and $\beta = 1$. Note that $\alpha > 0$ and $y^* > 0$ by Assumption S.6.1 and $\beta > 0$. Note also that:

$$\alpha Y^* = F(0,0)1 \leq F(c,0)1 = T(0) \leq Y^* = \beta Y^*$$

where $1 \in L^+$ is the sequence of ones, the first inequality follows because F increases in its first argument by Assumption 1. It follows that property (ii) holds.

The continuity of the mapping $c \mapsto V_t^c$ in the product topology follows from Theorem 6 in Marinacci and Montrucchio (2010). To conclude the proof we verify that the assumptions of their Theorem 6 hold. Note that F is continuous by Assumption 1; besides, it is Thompson and γ-subhomogeneous by Lemma S.6.1. For every consumption plan $c \in L_\infty$, their condition (17) becomes $\lim_{t\to+\infty} F(1,t)/t < 1$. This is satisfied because: (i) by Assumption S.6.1, there is some y such that $F(1,y) = y$, and (ii) by their Lemma 1, $F(1,t)/t$ is strictly decreasing. Again, we know that $\mathcal{R}^i$ is subhomogeneous by Lemma S.6.1. The continuity and monotonicity of $\mathcal{R}_t^i$ implies that: (i) $V_{t+1}^n \uparrow V_{t+1}$ implies $\mathcal{R}_t^i\left(V_{t+1}^n\right) \uparrow \mathcal{R}_t^i(V_{t+1})$ for every t, and (ii) $V^n \to V$ implies $\mathcal{R}_t^i\left(V_{t+1}^n\right) \to \mathcal{R}_t^i(V_{t+1})$. So the hypotheses for their Theorem 6 are satisfied. □

Lemma S.6.3. *Under Assumptions 1 and 2, $\succeq_i$ satisfies strong monotonicity.*

Proof. Suppose that $c \geq c'$, that is, $c(\sigma_t) \geq c'(\sigma_t)$ for all $\sigma_t \in \Sigma^t$ and $t \geq 1$. For all $T \geq 1$, we define

$$c^T(\sigma_t) = \begin{cases} c(\sigma_t) & t \leq T \\ c'(\sigma_t) & t > T \end{cases} \quad \text{for all } \sigma_t \in \Sigma^t. \tag{S.6.2}$$

Since F is increasing in its first argument, we have that for every $\sigma_T \in \Sigma^T$:

$$\begin{aligned} V_{\sigma_T}^i\left(c^T\right) &= F^i\left(c^T(\sigma_T), \mathcal{R}_T^i\left(V_{T+1}\left(c^T\right)\right)\right) \\ &= F^i\left(c^T(\sigma_T), \mathcal{R}_T^i\left(V_{T+1}(c')\right)\right) \\ &\geq F^i\left(c'(\sigma_T), \mathcal{R}_T^i\left(V_{T+1}(c')\right)\right) = V_{\sigma_T}^i(c'). \end{aligned} \tag{S.6.3}$$

Similarly, for all $\sigma_{T-1} \in \Sigma^{T-1}$, we have

$$\begin{aligned} V_{\sigma_{T-1}}^i\left(c^T\right) &= F^i\left(c^T(\sigma_{T-1}), \mathcal{R}_{T-1}^i\left(V_T\left(c^T\right)\right)\right) \\ &\geq F^i\left(c'(\sigma_{T-1}), \mathcal{R}_{T-1}^i\left(V_T(c')\right)\right) = V_{\sigma_{T-1}}^i(c'), \end{aligned}$$

where the inequality comes from (S.6.3), the monotonicity of F in both arguments by Assumption 1 and because $\mathcal{R}^i$ is increasing in continuation utilities by Assumption 2. By induction, we have

$$V^i_{\sigma_0}\left(c^T\right) \geq V^i_{\sigma_0}\left(c'\right).$$

Notice that as $T \to +\infty$, we have $c^T \to c$ in the product topology.[10] $V^i_{\sigma_0}$ is continuous in product topology, so we have

$$V^i_{\sigma_0}(c) = \lim_{T\to+\infty} V^i_{\sigma_0}\left(c^T\right) \geq V^i_{\sigma_0}\left(c'\right),$$

which proves that $\succeq_i$ satisfies weak monotonicity, that is, $c \geq c'$ implies $c \succeq_i c'$. To prove the strong monotonicity, we note that for any $k > 0$, we have

$$\begin{aligned} V^i_{\sigma_0}(c+k) &= F^i\left(c(\sigma_0)+k, \mathcal{R}^i_0\left(V_1(c+k)\right)\right) \\ &> F^i\left(c(\sigma_0), \mathcal{R}^i_0\left(V_1(c+k)\right)\right) \\ &\geq F^i\left(c(\sigma_0), \mathcal{R}^i_0\left(V_1(c)\right)\right) = V^i_{\sigma_0}(c), \end{aligned}$$

where the strict inequality comes from $F^i_1 > 0$ and the weak inequality comes from the weak monotonicity of $\succeq_i$ shown above. □

Remark 1. Lemma S.6.3 shows that V_0 is monotonic in c. From the proof, it is straightforward that V_t is monotonic in c for all $t \in \mathbb{T}$, that is, if $c \geq c'$ from time t on, then we have $V_t(c) \geq V_t(c')$.

Lemma S.6.4. *Under Assumptions 1, 2, S.6.1 and S.6.2, $\succeq_i$ satisfies weak convexity.*

Proof. Suppose that $c \neq c'$, $c \succeq_i c'$, and $\alpha \in (0,1)$. Again, we define c^T as in (S.6.2), and the concavity of F implies that for all $\sigma_T \in \Sigma^T$, we have

$$\begin{aligned} V^i_{\sigma_T}\left(\alpha c^T + (1-\alpha)c'\right) &= F^i\left(\alpha c^T(\sigma_T) + (1-\alpha)c'(\sigma_T), \mathcal{R}^i_T\left(V_{T+1}(c')\right)\right) \\ &\geq \alpha F^i\left(c^T(\sigma_T), \mathcal{R}^i_T\left(V_{T+1}(c')\right)\right) + (1-\alpha)F^i\left(c'(\sigma_T), \mathcal{R}^i_T\left(V_{T+1}(c')\right)\right) \\ &= \alpha V^i_{\sigma_T}\left(c^T\right) + (1-\alpha)V^i_{\sigma_T}(c') \end{aligned} \tag{S.6.4}$$

For all $\sigma_{T-1} \in \Sigma^{T-1}$, we have

$$\begin{aligned} &V^i_{\sigma_{T-1}}\left(\alpha c^T + (1-\alpha)c'\right) \\ =&F^i\left(\alpha c^T(\sigma_{T-1}) + (1-\alpha)c'(\sigma_{T-1}), \mathcal{R}^i_{T-1}\left(V_T\left(\alpha c^T + (1-\alpha)c'\right)\right)\right) \qquad \text{(S.6.5)} \\ \geq&F^i\left(\alpha c^T(\sigma_{T-1}) + (1-\alpha)c'(\sigma_{T-1}), \alpha\mathcal{R}^i_{T-1}\left(V^i_{\sigma_T}\left(c^T\right)\right) + (1-\alpha)\mathcal{R}^i_{T-1}\left(V^i_{\sigma_T}(c')\right)\right) \\ &\qquad \text{(S.6.6)} \end{aligned}$$

[10] That is, $\lim_{T\to+\infty} ||\pi_t\left(c^T\right) - \pi_t(c)||_\infty = 0$ for all $t \geq 1$, where $\pi_t(c)$ denotes the time-t projection of c.

$$\geq \alpha F^i\left(c^T\left(\sigma_{T-1}\right), \mathcal{R}^i_{T-1}\left(V^i_{\sigma_T}\left(c^T\right)\right)\right) + (1-\alpha) F^i\left(c'\left(\sigma_{T-1}\right), \mathcal{R}^i_{T-1}\left(V^i_{\sigma_T}\left(c'\right)\right)\right) \quad \text{(S.6.7)}$$

$$= \alpha V^i_{\sigma_{T-1}}\left(c^T\right) + (1-\alpha) V^i_{\sigma_{T-1}}\left(c'\right),$$

where (S.6.6) follows from (S.6.4) and the monotonicity and concavity of $\mathcal{R}^i_{T-1}$, and (S.6.7) comes from the concavity of F^i. By induction, we have

$$V^i_{\sigma_0}\left(\alpha c^T + (1-\alpha) c'\right) \geq \alpha V^i_{\sigma_0}\left(c^T\right) + (1-\alpha) V^i_{\sigma_0}\left(c'\right).$$

Again, since $c^T \to c$ in product topology and V^i is continuous in the product topology, we must have

$$V^i_{\sigma_0}\left(\alpha c + (1-\alpha) c'\right) \geq \alpha V^i_{\sigma_0}\left(c\right) + (1-\alpha) V^i_{\sigma_0}\left(c'\right),$$

which proves that $\succeq$ is convex. □

Proof of Theorem S.6.1

Proof. Let $B = \Sigma \times \mathbb{T}$, $\mathcal{B} = \mathcal{F} \otimes 2^{\mathbb{T}}$ and $\upsilon = \mathbb{P} \otimes \lambda$, where λ is the counting measure on $\left(\mathbb{T}, 2^{\mathbb{T}}\right)$. Conditions (1) and (2) in Theorem S.6.2 hold as $C_i = L^+_\infty(B, \mathcal{B}, \upsilon)$ for all $i \in I$. Condition (3) follows by our Assumption 3. Lemma S.6.2 shows preferences are continuous in the product topology. Since the product topology is coarser than the Mackey topology, it follows preferences are continuous in the Mackey topology as well and so (a) in condition (4) holds.[11] Finally, Lemmas S.6.3 and S.6.4 show preferences are strongly monotone and weakly convex, respectively, and so (b) and (c) in condition (4) hold. That (2) holds follows from Proposition 1 and the existence of a competitive equilibrium. □

S.7 Dynamics with Epstein–Zin Preferences

This appendix specializes the general market-selection dynamics to the Epstein–Zin preferences in Example 2. The goal is to construct a process $\hat{L}^{ij}_t$ that admits a decomposition of the form (8) and to compute the local survival index of agent j when the economy is close to the region in which agent i consumes the aggregate endowment. The main difficulty is that, as agent j becomes negligible, both j's consumption and continuation utilities converge to zero and so effective beliefs and effective discount rates are not well-defined. We therefore first obtain a novel closed-form solution to the Euler equations obtained in Proposition 1 jointly with the Arrow–Debreu budget constraint. The resulting representation allows us to characterize the limiting effective beliefs and modified discount rates needed to compute the local index. The proofs of this section can be found in Section S.8.

[11] See Lemma 2.48 in Aliprantis and Border (1999)

S.7.1 Closed-Form solutions

In what follows we consider the case $\gamma^i \neq 1$.[12] By Proposition 1 and Example 5, the Euler equation and the date-zero budget constraint are:

$$\frac{p(\sigma_t, s)}{p(\sigma_t)} = \beta \left(\frac{c^i(\sigma_t, s)}{c^i(\sigma_t)} \right)^{-\rho^i} \left(\frac{V^i_{\sigma_t, s}}{\mathcal{R}^i_t(V^i_{t+1})} . \right)^{\rho^i - \gamma^i} \pi^i(s) \tag{S.7.1}$$

$$\sum_{t=0}^{\infty} \sum_{\sigma \in \Sigma^t} p(\sigma_t) \times c^i(\sigma_t) = w(p, e^i) \tag{S.7.2}$$

where $w(p, e^i) \equiv \sum_{t=0}^{\infty} \sum_{\sigma \in \Sigma^t} p(\sigma_t) \times e^i(\sigma_t)$ is the date zero value of agent i's endowment.

Equations (S.7.1) and (S.7.2) are necessary conditions for optimality, but they do not exhaust all the restrictions imposed by the consumer's problem. Optimality also imposes a restriction on the relation between current consumption and continuation utility at every partial history. To see where this restriction comes from, fix a history σ_t and consider the following hypothetical problem: suppose that, at σ_t, Arrow–Debreu markets for all successor histories were reopened, and suppose that the continuation of the date-zero optimal plan played the role of the consumer's endowment from that point onward. The "wealth" associated with this hypothetical problem is simply the value, at the continuation prices induced by the original date-zero price system, of the remaining consumption promised by the date-zero optimal plan. Since Epstein–Zin continuation utility is homogeneous of degree one in continuation consumption and the hypothetical budget set is linear in this "wealth", the indirect utility of the auxiliary continuation problem is linear in that "wealth", with a proportionality factor $\phi^i_{\sigma_t}(p_{\sigma_t}) > 0$ that depends on the continuation price system but not on the level of "wealth". In this sense, $\phi^i_{\sigma_t}(p_{\sigma_t})$ is the marginal value of wealth at history σ_t.

The relevance of this hypothetical problem is that a date-zero optimal plan must remain optimal when viewed from any history σ_t in this counterfactual sense.[13] Hence, the continuation utility generated by the date-zero plan, $V^i_{\sigma_t}(c^i)$, must coincide with the indirect utility of the hypothetical continuation problem evaluated at the value of the continuation date-zero consumption that we denote by $w^i_{\sigma_t}(p_{\sigma_t}, c^i_{\sigma_t})$. Formally, at each σ_t :

$$V^i_{\sigma_t}(c^i) = \phi^i_{\sigma_t}(p_{\sigma_t})\, w^i_{\sigma_t}(p_{\sigma_t}, c^i_{\sigma_t}).$$

Applying Euler's theorem to the homogeneous Epstein–Zin utility along a consumption

[12] The case $\gamma^i = 1$ is analogous and is available from the authors upon request.

[13] If it did not, then the consumer could replace the continuation of the original plan after σ_t with another affordable continuation plan that delivers higher continuation utility, contradicting the optimality of the original date-zero plan.

process that satisfies the Euler equation, one obtains that the present discounted value of continuation consumption has a simple expression that depends directly on current consumption while future consumption enters only through the continuation utility. Indeed,

$$w^i_{\sigma_t}(p_{\sigma_t}, c^i_{\sigma_t}) = \frac{(c^i(\sigma_t))^{\rho^i}\left(V^i_{\sigma_t}(c^i)\right)^{1-\rho^i}}{1-\beta} \quad \text{for all } \sigma_t.$$

Substituting this expression into the continuation utility above and rearranging gives the following additional necessary conditions at each σ_t:

$$V^i_{\sigma_t}(c^i) = \left(\frac{\phi^i_{\sigma_t}(p)}{1-\beta}\right)^{1/\rho^i} c^i(\sigma_t) \tag{S.7.3}$$

We omit the proof to respect the page limit of the Supplementary Material; a complete proof is provided in the extended version of the supplementary material (Beker and Chen, 2026).

Proposition S.7.1. *If c^i solves consumer i's problem at σ_0, then there exists a positive process $\phi^i(\cdot|p)$ such that (c^i, ϕ^i) satisfies (S.7.1), (S.7.2) and (S.7.3).*

Let $\phi^i(\sigma_t|p) \equiv \phi^i_{\sigma_t}(p)$. We argue below that for any $\rho^i \neq 1$, $\left\{\phi^i(\sigma_t|p)\right\}_{\sigma_t \in \cup_\tau \Sigma^\tau}$ is a positive solution to the functional equation:[14]

$$\phi(\sigma_t) = \left[(1-\beta)^{\frac{1}{\rho^i}} + \beta^{\frac{1}{\rho^i}} p(\sigma_t)^{\frac{1-\rho^i}{\rho^i}} \left[\mathbb{E}_{\pi^i}\left(\frac{\phi(\sigma_t, s')\pi^i(s')}{p(\sigma^t, s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}\right]^{\frac{\gamma^i}{\rho^i}\frac{1-\rho^i}{1-\gamma^i}}\right]^{\frac{\rho^i}{1-\rho^i}} \tag{S.7.4}$$

For any $\gamma^i \neq 1$, define the consumption process recursively by

$$c^i(\sigma_0) = \left(p_0 + \sum_{t=1}^{\infty}\sum_{\sigma\in\Sigma^t} p(\sigma_t) \times \prod_{k=0}^{t-1} g^i(\sigma_k, s_{k+1}|p)\right)^{-1} w(p, e^i) \tag{S.7.5}$$

$$c^i(\sigma_t, s) = g^i(\sigma_t, s|p)c^i(\sigma_t) \qquad \text{for all } t \text{ and } \sigma_t \tag{S.7.6}$$

where

$$g^i(\sigma_t, s|p) \equiv \left(\beta\pi^i(s)\frac{p(\sigma_t)}{p(\sigma^t, s)}\right)^{\frac{1}{\rho^i}} \left(\frac{\left(\frac{\phi^i(\sigma_t, s|p)\pi^i(s)}{p(\sigma_t, s)}\right)^{\frac{1-\gamma^i}{\gamma^i}}}{\sum_{s'}\pi^i(s')\left(\frac{\phi^i(\sigma_t, s'|p)\pi^i(s')}{p(\sigma_t, s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}}\right)^{\frac{1}{\rho^i}\frac{\rho^i-\gamma^i}{1-\gamma^i}}$$

Proposition S.7.2. *Suppose c^i solves consumer i's problem with $c^i(\sigma_0) > 0$. Then (c^i, ϕ^i) satisfies (S.7.1), (S.7.2) and (S.7.3) if and only if (c^i, ϕ^i) satisfies (S.7.4), (S.7.5) and (S.7.6).*

The assumption that $c^i(\sigma_0)$ is positive is made here only to ensure that the denominator in (S.7.5) is finite. This proposition constructs an allocation satisfying the necessary first-order conditions (S.7.1), (S.7.2) and the additional condition (S.7.3). Since in any

[14] The functional equation can be found in Beker and Chen (2026) for the case of $\rho^i = 1$.

competitive equilibrium (c, p), prices are summable, the consumer's problem has a solution and $c^i(\sigma_0) > 0$ for every agent i, Proposition S.7.1 implies that (S.7.1), (S.7.2) and (S.7.3) are necessary conditions. Since Proposition S.7.2 shows the proposed allocation defined by (S.7.4), (S.7.5) and (S.7.6) is a solution to (S.7.1), (S.7.2) and (S.7.3), it follows that the equilibrium consumption allocation must be given by (S.7.5)–(S.7.6).

Theorem S.7.1. *Suppose (c, p) is a competitive equilibrium, then the equilibrium consumption allocation must be given by (S.7.4), (S.7.5) and (S.7.6).*

S.7.2 Dynamics

We next define the relative-consumption process that plays the role of $\hat{L}_t^{ij}$ in the Epstein–Zin case and derive its one-step dynamics. From the FOCs (S.7.1), we have that:

$$c^i(\sigma_t, s+t+1)^{-\rho^i} = \left(\beta^i \pi^i(s) \frac{p(\sigma_t)}{p(\sigma_t, s_{t+1})}\right)^{-1} \left(\frac{V^i_{\sigma_t, s_{t+1}}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\gamma^i - \rho^i} c^i(\sigma_t)^{-\rho^i}$$

Now consider any two agents i and j. Let $\hat{L}^{ij}(\sigma_t) \equiv \frac{c^i(\sigma_t)^{-\rho^i}}{c^j(\sigma_t)^{-\rho^j}}$. Then,

$$\hat{L}^{ij}(\sigma_t, s_{t+1}) = B^{ji}(\sigma_t, s_{t+1}) \times \tilde{D}^{ji}(\sigma_t, s_{t+1}) \times \hat{L}^{i,j}(\sigma_t)$$

where

$$B^{ji}(\sigma_t, s_{t+1}) \equiv \frac{\pi^j(s)}{\pi^i(s)} \frac{\left(\frac{V^j_{\sigma_t, s_{t+1}}}{\mathcal{R}^j_t(V^j_{t+1})}\right)^{-\gamma^j}}{\left(\frac{V^i_{\sigma_t, s_{t+1}}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{-\gamma^i}} \quad \text{and} \quad \tilde{D}^{ji}(\sigma_t, s_{t+1}) \equiv \frac{\beta^j}{\beta^i} \frac{\left(\frac{V^j_{\sigma_t, s_{t+1}}}{\mathcal{R}^j_t(V^j_{t+1})}\right)^{\rho^j}}{\left(\frac{V^i_{\sigma_t, s_{t+1}}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho^i}}$$

are the ratio of effective beliefs and a modified ratio of effective discount rates. In the limit when agent j consumes all, the modified ratio of discount rates coincides with the ratio of effective discount rates.

S.7.3 Convergence

Let $\bar{p}^i$ denote the equilibrium price system in the one-agent economy in which agent i consumes the aggregate endowment. Then,

$$\frac{\bar{p}^i(\sigma_t, s_{t+1})}{\bar{p}^i(\sigma_t)} = \beta \pi^i(s_{t+1}) \left(\frac{e(s_{t+1})}{e(s_t)}\right)^{-\rho^i} \left(\frac{\bar{V}^i_{s_{t+1}}}{\mathcal{R}^i(\bar{V}^i)}\right)^{\rho^i - \gamma^i}$$

The following proposition formalizes the sense in which, when agent i is locally dominant, the continuation price system is close to the one-agent-i price system.

Proposition S.7.3. *Suppose $V^i_{\sigma_t}(c)$ is continuous in the product topology. Fix $K \in \mathbb{N}$. For every $\epsilon > 0$, there exists $L > 0$ such that, if*

$$\max_{0 \le k \le K} \max_{\sigma_{t+k} \succeq \sigma_t} \max_{j \neq i} \widehat{L}^{ij}(\sigma_{t+k}) \le L,$$

then, for every successor history $\sigma_{t+k} \succeq \sigma_t$ with $0 \leq k \leq K$,

$$\left| \frac{p(\sigma_{t+k})}{p(\sigma_t)} - \frac{\bar{p}^i(\sigma_{t+k})}{\bar{p}^i(\sigma_t)} \right| < \epsilon.$$

Moreover, for every pair $s, s' \in S$,

$$\left| \frac{p(\sigma_t, s')}{p(\sigma_t, s)} - \frac{\bar{p}^i(\sigma_t, s')}{\bar{p}^i(\sigma_t, s)} \right| < \epsilon.$$

S.7.4 Local Relative Consumption Dynamics

We now derive the local dynamics of $\hat{L}_t^{ij}$ near the region in which agent i consumes the aggregate endowment. The following expression rewrites the expected log of the effective belief and the modified discount terms in a form that separates a normalized continuation-value component from the belief and discount-factor terms.

Proposition S.7.4. *For any agent i and partial history σ_t,*

$$\mathbb{E} \log \left[\pi^i(s) \left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})} \right)^{-\gamma^i} \left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})} \right)^{\rho^i} \beta^i \right] = \left(1 - \tfrac{1-\rho^i}{1-\gamma^i}\right) \times \mathbb{E} \log \left(\frac{\pi^i(s) (V^i_{\sigma_t,s})^{1-\gamma^i}}{\mathbb{E}_{\pi^i}((V^i_{t+1})^{1-\gamma^i})} \right) + \tfrac{1-\rho^i}{1-\gamma^i} \times \mathbb{E} \log \pi^i(s) + \log \beta^i$$

For any $\rho^j, \gamma^j > 0$ with $\gamma^i, \gamma^j \neq 1$, we define the index:

$$\mathcal{S}_i^j = \left(1 - \tfrac{1-\rho^j}{1-\gamma^j}\right) \times \mathbb{E} \log \left(\frac{\pi^i(s) \left(\frac{\pi^j(s)}{\pi^i(s)}\right)^{\frac{1}{\gamma^j}} \left(\frac{\phi^j(s|\bar{p}^i)}{\phi^i(s|\bar{p}^i)}\right)^{\frac{1-\gamma^j}{\gamma^j}} (\bar{V}^i_s)^{\gamma^i \frac{1-\gamma^j}{\gamma^j}}}{\sum_{s'} \pi^i(s') \left(\frac{\pi^j(s')}{\pi^i(s')}\right)^{\frac{1}{\gamma^j}} \left(\frac{\phi^j(s'|\bar{p}^i)}{\phi^i(s'|\bar{p}^i)}\right)^{\frac{1-\gamma^j}{\gamma^j}} (\bar{V}^i_{s'})^{\gamma^i \frac{1-\gamma^j}{\gamma^j}}} \right) + \tfrac{1-\rho^j}{1-\gamma^j} \times \mathbb{E} \log \pi^j(s) + \log \beta^j \tag{S.7.7}$$

The index $\mathcal{S}_i^j$ is the asymptotic expected log growth rate of agent j's consumption share, powered to $-\rho^j$, when the economy is near the region in which agent i consumes the aggregate endowment. If $\mathcal{S}_i^j < \mathcal{S}_i^i$, then agent j's relative consumption tends to decline near the i-dominated region; if $\mathcal{S}_i^j > \mathcal{S}_i^i$ then agent j tends to recover.

By Proposition S.7.4, it suffices to show that

$$\mathbb{E} \log \left(\frac{\pi^j(s) \left(V^j_{\sigma_t,s}\right)^{1-\gamma^j}}{\mathbb{E}_{\pi^j}\left(\left(V^j_{t+1}\right)^{1-\gamma^j}\right)} \right) \approx \mathbb{E} \log \left(\frac{\pi^i(s) \left(\frac{\pi^j(s)}{\pi^i(s)}\right)^{\frac{1}{\gamma^j}} \left(\frac{\phi^j(s|\bar{p}^i)}{\phi^i(s|\bar{p}^i)}\right)^{\frac{1-\gamma^j}{\gamma^j}} (\bar{V}^i_s)^{\gamma^i \frac{1-\gamma^j}{\gamma^j}}}{\sum_{s'} \pi^i(s') \left(\frac{\pi^j(s')}{\pi^i(s')}\right)^{\frac{1}{\gamma^j}} \left(\frac{\phi^j(s'|\bar{p}^i)}{\phi^i(s'|\bar{p}^i)}\right)^{\frac{1-\gamma^j}{\gamma^j}} (\bar{V}^i_{s'})^{\gamma^i \frac{1-\gamma^j}{\gamma^j}}} \right) \text{ for all } j$$

Now we are ready to show the main result of this section.

Theorem S.7.2. *Suppose that for every agent h, c^h is given by (S.7.5) and (S.7.6) and the map $\phi^h(\cdot|p)$ is continuous with respect to prices in the product topology. For every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ and $\underline{L} > 0$ such that*

$$\max_{0 \leq k \leq K} \max_{\sigma_{t+k} \succeq \sigma_t} \max_{j \neq i} \widehat{L}^{ij}(\sigma_{t+k}) \leq \underline{L} \Rightarrow \left| \mathbb{E}\left[\log \left(B^{ji}(\sigma_t, s_{t+1}) \times \tilde{D}^{ji}(\sigma_t, s_{t+1}) \right) \right] - \left(\mathcal{S}_i^j - \mathcal{S}_i^i \right) \right| < \epsilon$$

S.8 Proofs of Section S.7

The proof of Proposition S.7.2 uses the following result.[15]

Lemma S.8.1. *Suppose* (c^i, ϕ^i) *satisfies* (S.7.4), (S.7.5) *and* (S.7.6). *Then,*

$$\left(\mathbb{E}_{\pi^i}\left[\left(\left(\frac{\phi^i(\sigma_{t+1}|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i_{t+1}\right)^{1-\gamma^i}\right]\right)^{\frac{1}{1-\gamma^i}} = \left(\frac{\beta p(\sigma_t)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_t)\left(\left(\mathbb{E}_{\pi^i}\left[\left(\frac{\pi^i(s_{t+1})\phi^i(\sigma_{t+1}|p)}{p(\sigma_{t+1})}\right)^{\frac{1-\gamma^i}{\gamma^i}}\right]\right)^{\frac{\gamma^i}{\rho^i}}\right)^{\frac{1}{1-\gamma^i}} \tag{S.8.1}$$

$$\left(\frac{\left(\left(\frac{\phi^i(\sigma_t,s|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_t,s)\right)^{1-\gamma^i}}{\mathbb{E}_{\pi^i}\left(\left(\frac{\phi^i(\sigma_{t+1}|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_{t+1})\right)^{1-\gamma^i}}\right)^{\frac{1}{1-\gamma^i}} = \left[\frac{\left(\frac{\pi^i(s)\phi^i(\sigma_t,s|p)}{p(\sigma^t,s)}\right)^{\frac{1-\gamma^i}{\gamma^i}}}{\mathbb{E}_{\pi^i}\left(\frac{\pi^i(s_{t+1})\phi^i(\sigma_{t+1}|p)}{p(\sigma_{t+1})}\right)^{\frac{1-\gamma^i}{\gamma^i}}}\right]^{\frac{1}{1-\gamma^i}} \tag{S.8.2}$$

(S.8.3)

S.8.1 Proof of Proposition S.7.2

Proof. Suppose $c^i_0 > 0$. Without loss of generality, let $p_0 = 1$. First we show that (c^i, ϕ^i) given by (S.7.4), (S.7.5) and (S.7.6) satisfies equations (S.7.1), (S.7.2) and (S.7.3).

To show (S.7.3) holds, it suffices to show that $\left(\frac{\phi^i(\sigma_t|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_t)$ is a solution to the Epstein–Zin recursion. We consider the cases of $\rho^i = 1$ and $\rho^i \neq 1$ separately.

- For the case $\rho^i \neq 1$

$$\begin{aligned}
\phi^i(\sigma_t|p)^{\frac{1-\rho^i}{\rho^i}}\left(\frac{c^i(\sigma_t)}{(1-\beta)^{\frac{1}{\rho^i}}}\right)^{1-\rho^i} &= \left((1-\beta)^{\frac{1}{\rho^i}} + \beta^{\frac{1}{\rho^i}} p(\sigma_t)^{\frac{1-\rho^i}{\rho^i}}\left(\mathbb{E}_{\pi^i}\left[\left(\frac{\pi^i(s')\phi^i(\sigma_t,s'|p)}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}\right]\right)^{\frac{\gamma^i}{\rho^i}\frac{1-\rho^i}{1-\gamma^i}}\right)\left(\frac{c^i(\sigma_t)}{(1-\beta)^{\frac{1}{\rho^i}}}\right)^{1-\rho^i} \\
&= (1-\beta)c^i(\sigma_t)^{1-\rho^i} + \beta\left(\frac{\beta p(\sigma_t)}{1-\beta}\right)^{\frac{1-\rho^i}{\rho^i}} c^i(\sigma_t)^{1-\rho^i}\left(\mathbb{E}_{\pi^i}\left[\left(\frac{\pi^i(s')\phi^i(\sigma_t,s'|p)}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}\right]\right)^{\frac{\gamma^i}{\rho^i}\frac{1-\rho^i}{1-\gamma^i}} \\
&= (1-\beta)c^i(\sigma_t)^{1-\rho^i} + \beta\left(\left(\mathbb{E}_{\pi^i}\left[\left(\left(\frac{\phi^i(\sigma_t,s'|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_t,s')\right)^{1-\gamma^i}\right]\right)^{\frac{1}{1-\gamma^i}}\right)^{1-\rho^i}
\end{aligned}$$

  where the first equality follows from (S.7.4) and the last one from (S.7.4). It follows that

$$\left(\frac{\phi^i(\sigma_t|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_t) = \left[(1-\beta)c^i(\sigma_t)^{1-\rho^i} + \beta\left(\mathbb{E}_{\pi^i}\left[\left(\left(\frac{\phi^i(\sigma_t,s'|p)}{1-\beta}\right)^{\frac{1}{\rho^i}} c^i(\sigma_t,s')\right)^{1-\gamma^i}\right]\right)^{\frac{1-\rho^i}{1-\gamma^i}}\right]^{\frac{1}{1-\rho^i}}$$

- For the case $\rho^i = 1$,

[15] See the proof in Beker and Chen (2026).

$$\frac{\phi^i(\sigma_t|p)}{1-\beta}c^i(\sigma_t) = c^i(\sigma_t)\left(\frac{\beta p(\sigma_t)}{1-\beta}\right)^{\beta}\left(\mathbb{E}_{\pi^i}\left[\left(\frac{\pi^i(s')\phi^i(\sigma_t,s'|p)}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}\right]\right)^{\frac{\beta\gamma^i}{1-\gamma^i}}$$
$$= c^i(\sigma_t)^{1-\beta}\left(\left(\frac{\beta p(\sigma_t)}{1-\beta}\right)c^i(\sigma_t)\left(\left[\mathbb{E}_{\pi^i}\left(\frac{\pi^i(s')\phi^i(\sigma_t,s'|p)}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}\right]^{\gamma^i}\right)^{\frac{1}{1-\gamma^i}}\right)^{\beta}$$
$$= c^i(\sigma_t)^{1-\beta}\left(\mathbb{E}_{\pi^i}\left[\left(\left(\frac{\phi^i(\sigma_t,s'|p)}{1-\beta}\right)^{\frac{1}{\rho^i}}c^i(\sigma_t,s')\right)^{1-\gamma^i}\right]\right)^{\frac{\beta}{1-\gamma^i}}$$

where the first equality follows from (S.7.4) and the last one from (S.7.4).

To show it satisfies the budget constraint (S.7.2), note that as we assumed $c_0^i > 0$, then:

$$p_0 + \sum_{t=1}^{\infty}\sum_{\sigma\in\Sigma^t} p(\sigma_t)\left(\prod_{k=0}^{t-1} g^i(\sigma_k, s_{k+1}|p)\right) = \frac{w(p,e^i)}{c_0^i} < \infty$$

Thus,

$$\sum_{t=0}^{\infty}\sum_{\sigma\in\Sigma^t} p(\sigma_t)\times c^i(\sigma_t) = \left[p_0 + \sum_{t=1}^{\infty}\sum_{\sigma\in\Sigma^t} p(\sigma_t)\left(\prod_{k=0}^{t-1} g^i(\sigma_k, s_{k+1}|p)\right)\right]c_0^i = w(p,e^i)$$

Finally, we show the Euler Equation (S.7.1) holds. Note that:

$$\beta\pi^i(s)\left(\frac{c^i(\sigma_t,s)}{c^i(\sigma_t)}\right)^{-\rho^i}\left(\frac{V^i_{\sigma_t,s}(c^i)}{\mathcal{R}^i_t(V^i_{t+1})}.\right)^{\rho^i-\gamma^i} = \beta\pi^i(s)\left(\beta\pi^i(s)\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{-1}\left(\frac{\left(\frac{\phi^i(\sigma_t,s|p)\pi^i(s)}{p(\sigma_t,s)}\right)^{\frac{1-\gamma^i}{\gamma^i}}}{\sum_{s'}\pi^i(s')\left(\frac{\phi^i(\sigma_t,s'|p)\pi^i(s')}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}}\right)^{-\frac{\rho^i-\gamma^i}{1-\gamma^i}}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho^i-\gamma^i}$$
$$= \left(\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{-1}\left(\frac{\left(\left(\frac{\phi^i(\sigma_t,s|p)}{1-\beta}\right)^{\frac{1}{\rho^i}}c^i(\sigma_t,s)\right)^{1-\gamma^i}}{\mathbb{E}_{\pi^i}\left(\left(\frac{\phi^i(\sigma_{t+1}|p)}{1-\beta}\right)^{\frac{1}{\rho^i}}c^i(\sigma_{t+1})\right)^{1-\gamma^i}}\right)^{\frac{1}{1-\gamma^i}}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{-(\rho^i-\gamma^i)}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho^i-\gamma^i}$$
$$= \left(\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{-1}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{-(\rho^i-\gamma^i)}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho^i-\gamma^i} = \left(\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{-1}$$

where the second step follows by (S.8.2) and the third one as we have shown (S.7.3) holds.

Now we argue the converse is true: if (c^i, ϕ^i) satisfies (S.7.1), (S.7.2) and (S.7.3), then it satisfies (S.7.4), (S.7.5) and (S.7.6). Note that equation (S.7.1) can be rewritten as:

$$\left(\frac{c^i(\sigma_t,s)}{c^i(\sigma_t)}\right)^{\rho^i} = \beta\left(\frac{p(\sigma^t,s)}{p(\sigma_t)}\right)^{-1}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho^i-\gamma^i}\pi^i(s)$$

and so

$$\frac{c^i(\sigma_t,s)}{c^i(\sigma_t)} = \left(\beta\pi^i(s)\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{\frac{1}{\rho^i}}\left(\frac{\left(V^i_{\sigma_t,s}\right)^{1-\gamma^i}}{\sum_{s'}\pi^i(s')\left(V^i_{\sigma_t,s'}(c^i)\right)^{1-\gamma^i}}\right)^{\frac{1}{\rho^i}\frac{\rho^i-\gamma^i}{1-\gamma^i}}$$

$$= \left(\beta\pi^i(s)\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{\frac{1}{\rho^i}} \left(\frac{\left(\phi^i(\sigma_t,s|p)^{\frac{1}{\rho^i}}\, c^i(\sigma_t,s)\right)^{1-\gamma^i}}{\sum_{s'}\pi^i(s')\left(\phi^i(\sigma_t,s'|p)^{\frac{1}{\rho^i}}\, c^i(\sigma_t,s')\right)^{1-\gamma^i}}\right)^{\frac{1}{\rho^i}\frac{\rho^i-\gamma^i}{1-\gamma^i}} \tag{S.8.4}$$

To compute the second parenthesis, take ratios between any two states s and s' in (S.8.4):

$$\frac{c^i(\sigma_t,s')}{c^i(\sigma_t,s)} = \left(\frac{p(\sigma^t,s')}{p(\sigma^t,s)}\right)^{-\frac{1}{\rho^i}} \left(\frac{\left(\phi^i(\sigma_t,s'|p)\right)^{\frac{1}{\rho^i}} c^i(\sigma_t,s')}{\left(\phi^i(\sigma_t,s|p)\right)^{\frac{1}{\rho^i}} c^i(\sigma_t,s)}\right)^{\frac{\rho^i-\gamma^i}{\rho^i}} \left(\frac{\pi^i(s')}{\pi^i(s)}\right)^{\frac{1}{\rho^i}}$$

and solve for $\frac{c^i(\sigma_t,s')}{c^i(\sigma_t,s)}$:

$$\frac{c^i(\sigma_t,s')}{c^i(\sigma_t,s)} = \left(\frac{p(\sigma^t,s')}{p(\sigma^t,s)}\right)^{-\frac{1}{\gamma^i}} \left(\left(\frac{\phi^i(\sigma_t,s'|p)}{\phi^i(\sigma_t,s|p)}\right)^{\frac{1}{\rho^i}}\right)^{\frac{\rho^i-\gamma^i}{\gamma^i}} \left(\frac{\pi^i(s')}{\pi^i(s)}\right)^{\frac{1}{\gamma^i}}$$

Next, notice that

$$\left(\frac{\phi^i(\sigma_t,s'|p)}{\phi^i(\sigma_t,s|p)}\right)^{\frac{1}{\rho^i}} \frac{c^i(\sigma_t,s')}{c^i(\sigma_t,s)} = \left(\frac{p(\sigma^t,s')}{p(\sigma^t,s)}\right)^{-\frac{1}{\gamma^i}} \left(\frac{\phi^i(\sigma_t,s'|p)}{\phi^i(\sigma_t,s|p)}\right)^{\frac{1}{\gamma^i}} \left(\frac{\pi^i(s')}{\pi^i(s)}\right)^{\frac{1}{\gamma^i}}$$

Then,

$$\begin{aligned}\frac{\sum_{s'}\pi^i(s')\left(\phi^i(\sigma_t,s'|p)^{\frac{1}{\rho^i}}\, c^i(\sigma_t,s')\right)^{1-\gamma^i}}{\left(\phi^i(\sigma_t,s|p)^{\frac{1}{\rho^i}}\, c^i(\sigma_t,s)\right)^{1-\gamma^i}} &= \sum_{s'}\pi^i(s')\left(\left(\frac{\phi^i(\sigma_t,s'|p)}{\phi^i(\sigma_t,s|p)}\right)^{\frac{1}{\rho^i}}\frac{c^i(\sigma_t,s')}{c^i(\sigma_t,s)}\right)^{1-\gamma^i} \\ &= \frac{\sum_{s'}\pi^i(s')\left(\pi^i(s')\frac{\phi^i(\sigma_t,s'|p)}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}}{\left(\pi^i(s)\frac{\phi^i(\sigma_t,s|p)}{p(\sigma_t,s)}\right)^{\frac{1-\gamma^i}{\gamma^i}}}\end{aligned} \tag{S.8.5}$$

If we now plug (S.8.5) into (S.8.4), we obtain:

$$\frac{c^i(\sigma_t,s)}{c^i(\sigma_t)} = \left(\beta\pi^i(s)\frac{p(\sigma_t)}{p(\sigma^t,s)}\right)^{\frac{1}{\rho^i}} \left(\frac{\left(\pi^i(s)\frac{\phi^i(\sigma_t,s|p)}{p(\sigma_t,s)}\right)^{\frac{1-\gamma^i}{\gamma^i}}}{\sum_{s'}\pi^i(s')\left(\pi^i(s')\frac{\phi^i(\sigma_t,s'|p)}{p(\sigma_t,s')}\right)^{\frac{1-\gamma^i}{\gamma^i}}}\right)^{\frac{1}{\rho^i}\frac{\rho^i-\gamma^i}{1-\gamma^i}}$$

and it follows that (S.7.6) holds. To show (S.7.5), note that, since c^i satisfies (S.7.2), then

$$w(p,e^i) = \sum_{t=0}^{\infty}\sum_{\sigma\in\Sigma^t} p(\sigma_t)\times c^i(\sigma_t) = \left[p_0 + \sum_{t=1}^{\infty}\sum_{\sigma\in\Sigma^t} p(\sigma_t)\times\prod_{k=0}^{t-1} g^i(\sigma_k,s_{k+1}|p)\right]c_0^i \qquad \square$$

S.8.2 Proof of Proposition S.7.3

See Beker and Chen (2026).

S.8.3 Proof of Proposition S.7.4

$$\begin{aligned}\mathbb{E}\log\left[\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{-\gamma^i}\left(\frac{V^i_{\sigma_t,s}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho^i}\beta^i\pi^i(s)\right] &= \frac{\rho^i-\gamma^i}{1-\gamma^i}\,\mathbb{E}\log\left(\frac{\pi^i(s)\left(V^i_{\sigma_t,s}\right)^{1-\gamma^i}}{\mathbb{E}_{\pi^i}\left(\left(V^i_{t+1}\right)^{1-\gamma^i}\right)}\right) + \frac{1-\rho^i}{1-\gamma^i}\mathbb{E}\log\pi^i(s) + \log\beta^i \\ &= \left(1-\frac{1-\rho^i}{1-\gamma^i}\right)\times\mathbb{E}\log\left(\frac{\pi^i(s)\left(V^i_{\sigma_t,s}\right)^{1-\gamma^i}}{\mathbb{E}_{\pi^i}\left(\left(V^i_{t+1}\right)^{1-\gamma^i}\right)}\right) + \frac{1-\rho^i}{1-\gamma^i}\times\mathbb{E}\log\pi^i(s) + \log\beta^i\end{aligned}$$

S.8.4 Proof of Theorem S.7.2

Proof. Note that (S.8.2) implies:

$$\frac{\pi^j(s)\left(V^j_{\sigma_t,s}\right)^{1-\gamma^j}}{\mathbb{E}_{\pi^j}\left[\left(V^j_{t+1}\right)^{1-\gamma^j}\right]} = \frac{\pi^j(s)^{\frac{1}{\gamma^j}}\phi^j(\sigma_t,s|p)^{\frac{1-\gamma^j}{\gamma^j}}p(\sigma_t,s)^{-\frac{1-\gamma^j}{\gamma^j}}}{\sum_{s'}\pi^j(s')^{\frac{1}{\gamma^j}}\phi^j(\sigma_t,s'|p)^{\frac{1-\gamma^j}{\gamma^j}}p(\sigma_t,s')^{-\frac{1-\gamma^j}{\gamma^j}}} = \frac{1}{\sum_{s'}\left(\frac{\pi^j(s')}{\pi^j(s)}\right)^{\frac{1}{\gamma^j}}\left(\frac{\phi^j(\sigma_t,s'|p)}{\phi^j(\sigma_t,s|p)}\right)^{\frac{1-\gamma^j}{\gamma^j}}\left(\frac{p(\sigma_t,s')}{p(\sigma_t,s)}\right)^{-\frac{1-\gamma^j}{\gamma^j}}}$$

Proposition S.7.3 and the continuity of $\phi^j(\cdot|p)$ in product topology implies:

$$\frac{\phi^j(\sigma_t,s'|p)^{\frac{1-\gamma^j}{\gamma^j}}p(\sigma_t,s')^{-\frac{1-\gamma^j}{\gamma^j}}}{\phi^j(\sigma_t,s|p)^{\frac{1-\gamma^j}{\gamma^j}}p(\sigma_t,s)^{-\frac{1-\gamma^j}{\gamma^j}}} \to \frac{\phi^j(s'|\bar{p}^i)^{\frac{1-\gamma^j}{\gamma^j}}\pi^i(s')^{-\frac{1-\gamma^j}{\gamma^j}}e(s')^{\rho^i\frac{1-\gamma^j}{\gamma^j}}\left(\bar{V}^i_{s'}\right)^{-(\rho^i-\gamma^i)\frac{1-\gamma^j}{\gamma^j}}}{\phi^j(s|\bar{p}^i)^{\frac{1-\gamma^j}{\gamma^j}}\pi^i(s)^{-\frac{1-\gamma^j}{\gamma^j}}e(s)^{\rho^i\frac{1-\gamma^j}{\gamma^j}}\left(\bar{V}^i_{s}\right)^{-(\rho^i-\gamma^i)\frac{1-\gamma^j}{\gamma^j}}}$$

It follows that

$$\frac{\pi^j(s)\left(V^j_{\sigma_t,s}\right)^{1-\gamma^j}}{\mathbb{E}_{\pi^j}\left[\left(V^j_{t+1}\right)^{1-\gamma^j}\right]} \to \frac{\pi^i(s)\left(\frac{\pi^j(s)}{\pi^i(s)}\right)^{\frac{1}{\gamma^j}}\phi^j(s|\bar{p}^i)^{\frac{1-\gamma^j}{\gamma^j}}e(s)^{\rho^i\frac{1-\gamma^j}{\gamma^j}}\left(\phi^i(s|\bar{p}^i)^{\frac{1}{\rho^i}}e(s)\right)^{-(\rho^i-\gamma^i)\frac{1-\gamma^j}{\gamma^j}}}{\sum_{s'}\pi^i(s')\left(\frac{\pi^j(s')}{\pi^i(s')}\right)^{\frac{1}{\gamma^j}}\phi^j(s'|\bar{p}^i)^{\frac{1-\gamma^j}{\gamma^j}}e(s')^{\rho^i\frac{1-\gamma^j}{\gamma^j}}\left(\phi^i(s'|\bar{p}^i)^{\frac{1}{\rho^i}}e(s')\right)^{-(\rho^i-\gamma^i)\frac{1-\gamma^j}{\gamma^j}}}$$

$$= \frac{\pi^i(s)\left(\frac{\pi^j(s)}{\pi^i(s)}\right)^{\frac{1}{\gamma^j}}\left(\frac{\phi^j(s|\bar{p}^i)}{\phi^i(s|\bar{p}^i)}\right)^{\frac{1-\gamma^j}{\gamma^j}}\left(\bar{V}^i_{s}\right)^{\gamma^i\frac{1-\gamma^j}{\gamma^j}}}{\sum_{s'}\pi^i(s')\left(\frac{\pi^j(s')}{\pi^i(s')}\right)^{\frac{1}{\gamma^j}}\left(\frac{\phi^j(s'|\bar{p}^i)}{\phi^i(s'|\bar{p}^i)}\right)^{\frac{1-\gamma^j}{\gamma^j}}\left(\bar{V}^i_{s'}\right)^{\gamma^i\frac{1-\gamma^j}{\gamma^j}}}$$

Since the limiting ratios are positive for all $s \in S$, for $\underline{\mathrm{L}}$ small enough, all terms are bounded away from zero; hence the log is uniformly continuous on the relevant compact set and the result holds. □

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