If You're NOT So Smart, Why Are You Rich? Robust Market Selection with General Recursive Preferences*

Pablo F. Beker[†] Jaden Yang Chen[‡]

October 28, 2023

Abstract

This paper provides a unified framework to study the long-run dynamics of competitive equilibrium consumption allocations in stochastic exchange economies with complete markets and multiple consumers whose preferences belong to a general class of recursive preferences, encompassing both expected-utility and various common non-expected utility preferences. We provide a characterization of the long-run consumption dynamics and apply it to study the robustness of the *market selection hypothesis* that market favors consumers with more accurate beliefs. In sharp contrast to the results for the class of expected utility studied by Sandroni (2000) and Blume and Easley (2006), we show that the existence of multiple survivors is the robust long-run outcome for our general class of preferences. Our results imply there is an inherent tension between two broadly held tenets in economics and finance, that markets allocate resources efficiently and equilibrium allocations can eventually be described as a rational expectations equilibrium.

Keywords: Market selection hypothesis, rational expectations, belief heterogeneity, heterogeneous consumers, recursive preferences, non-expected utility, robustness

*Acknowledgments to be added later.
[†]University of Warwick, E-mail: pablo.beker@warwick.ac.uk
[‡]University of North Carolina at Chapel Hill, E-mail: yangch@unc.edu

1 Introduction

A long-standing tenet in economics is that agents who do not make correct predictions are driven out of the market, in the sense that they lose their wealth at the hand of those that make more correct predictions. This tenet, put forward by authors like Alchian (1950), Friedman (1953) and Cootner (1964), is known as the *market selection hypothesis* (henceforth, MSH). The MSH is very influential. It is often invoked to justify the use of rational expectations equilibrium as a solution concept in modern economics because it implies that asset prices eventually reflect only the beliefs of agents making correct predictions.

Sandroni (2000) and Blume and Easley (2006) study formally the market selection hypothesis in a general equilibrium framework.[1] They find the MSH holds in a complete market with bounded endowment when consumers have subjective *expected utility* (SEU hereafter) preferences.[2] Specifically, they show that among consumers with the same discount rate, survival depends only on the accuracy of beliefs. There has already been some progress on relaxing the SEU assumption. Condie (2008) consider the cases of max-min preferences and shows that only consumers with SEU and correct beliefs survive.[3] Guerdjikova and Sciubba (2015) show that consumers with smooth ambiguity aversion preferences and correct beliefs can drive SEU maximizers out of the market. Dindo (2019) and Borovička (2020) show that consumers with homothetic Epstein-Zin preferences and incorrect beliefs can survive in the presence of consumers with correct beliefs. These findings suggest there might be a tension between two broadly held tenets in economics and finance: (a) markets allocate resources efficiently and (b) equilibrium allocations can be described as a rational expectations equilibrium. However, these recent papers do not provide a definite answer as they only consider a particular set of preferences and the MSH could be restored once one allows for other preferences.

In this paper, we provide a unified framework to study the MSH when consumers' preferences belong to a general class of *recursive preferences*, nesting both expected-utility and many common non-expected utility preferences as special cases. More formally, we allow consumers' preferences to take the following recursive form

$$V_t = F\left(c_t, \mathcal{R}_t\left(V_{t+1}\right)\right),$$

where F is the time aggregator which describes the attitude toward temporal resolution of uncertainty, and $\mathcal{R}$ is the certainty equivalent aggregator which describes the attitude toward the uncertainty in continuation values. By appropriately choosing F and $\mathcal{R}$, our framework can nest almost all

[1] Blume and Easley (1992) were the first to study the MSH in a market clearing model but they do not allow for intertemporal optimization. Cogley and Sargent (2009) study the quantitative effect of the MSH on asset prices.

[2] Some alternative setups have been studied, for example unbounded endowments (Yan (2008), Cvitanić and Malamud (2011) and Kogan et al. (2006, 2017)), incomplete markets (Blume and Easley (2006), Beker and Chattopadhyay (2010), Coury and Sciubba (2012), Cao (2017) and Cogley et al. (2014)), continuum of agents (Massari, 2019) and asymmetric information (Mailath and Sandroni, 2003).

[3] Da Silva (2011) and Easley and Yang (2015) consider the cases of variational and loss aversion preferences, respectively, but also fail to identify a persistent impact on market outcomes.

preferences studied in the MSH literature, and many other interesting deviations from SEU that haven't been explored before. Our general analysis shows the MSH is *not* robust both globally and locally in a sense that will be made precise later. Furthermore, our analysis implies that the robust long-run outcome with general preferences is the existence of multiple survivors. Therefore, our paper makes evident there is actually an inherent tension between the two aforementioned tenets in economics and finance.

In what follows, we describe our approach and results in greater detail. We consider an infinite-horizon exchange economy similar to Sandroni (2000) and Blume and Easley (2006) extended to allow for general recursive preferences. We begin our analysis by showing the existence of a competitive equilibrium, which is based on the existence theorem due to Bewley (1969) that requires preferences to be strongly monotone, weakly convex and continuous in the Mackey topology. We assume monotonicity and concavity of the aggregators to argue that the generated recursive preferences are strongly monotone and weakly convex, and we adapt the arguments in Marinacci and Montrucchio (2010) to prove the desired continuity property.

Afterwards we provide our characterization of the long-run consumption dynamics. As in Sandroni (2000) and Blume and Easley (2006), we study the evolution of consumption by studying the dynamics of the ratio of marginal utilities of consumption stemming from the first-order condition of the consumer's problem. We show that in any competitive equilibrium, the growth rate of any two consumers' marginal utility ratio at date t depends on the ratio of two terms: (i) the consumer's marginal value of the certainty equivalent aggregator of future utility, F_2, which is referred to as the *effective discount rate*, and (ii) the consumers' marginal certainty equivalent value of next period utilities, $\nabla \mathcal{R}_t$, which is referred to as the *effective belief*. When consumers have SEU preferences, the effective discount rate and beliefs become the standard discount rate and subjective beliefs, so the dynamics of marginal utility ratios are relatively simple as both objects are *exogenously* given. With general recursive preferences, instead, both effective beliefs and effective discount rate are *endogenously* determined and can depend on consumers' equilibrium consumption.

To tackle this technical difficulty, this paper adopts a local approach that consists in studying the (exogenous) dynamics of the marginal utility ratio when some consumer consumes the aggregate endowment and argue that it approximates the original dynamics. We define a local domination relation by saying that i *locally dominates* j if the expected log of consumer i's effective discount rate and beliefs is larger than that of consumer j when i consumes all the endowment. We show that when i locally dominates j, the log ratio of their marginal utilities can be approximated by a local supermartingale, which allows us to show that when i consumes almost all the endowment, there is a positive probability that she dominates the market, i.e., is the unique consumer who survives in the limit. Under the assumption that each such "one-consumes-all" neighborhood is visited with positive probability, we characterize the long-run dynamics in terms of our local domination relationship. We first consider the case of two consumers in Theorem 2. In sharp contrast with the expected-utility case where, except for tie cases, the consumer with more accurate beliefs is the unique survivor, Theorem 2 shows that a variety of patterns can emerge with general utility. We construct examples of economies where either (a) each consumer survives with positive probability or (b) both consumers co-exist with probability one or (c) some consumer with incorrect beliefs is the unique survivor. Afterwards, we extend the discussion to arbitrary number of consumers. Theorem 3 and 4 provide our characterization for the case of multiple consumers. It is worth noting that with general preferences, the survival patterns with multiple consumers can't be considered as a simple extension of the two-consumer case. To illustrate this point, Example 15 shows that a consumer vanishes almost surely in a two-consumer economy but survives almost surely in a larger economy.

We later apply our characterization to study the global and local robustness of the MSH. We restrict attention to the class of preferences generated by identical separable time aggregators, so that preferences are only determined by the certainty equivalent $\mathcal{R}$. We first show that the MSH is *globally non-robust* in Theorem 5. That is to say, there doesn't exist any $\mathcal{R}$ that always dominates the market. Furthermore, we show that there exist multiple "robust survivors", i.e., preferences that survive with positive probability in the presence of any possible preferences. Consequently, even though Sandroni (2000) and Blume and Easley (2006) show there is a unique preference in the SEU class that dominates the market, we argue that such dominating preference doesn't exist in the general class of preferences we study. Next, we further show that the MSH is *locally non-robust* as well in Theorem 7. That is to say, there doesn't exist any $\mathcal{R}$ that always dominates the market in the presence of sufficiently similar preferences. Furthermore, we show that there exist multiple "locally robust survivors", i.e., preferences that survive with positive probability in the presence of any sufficiently similar preferences. Perhaps surprisingly, we show that every preference that is not effectively SEU, in a sense that will be formalized later, is a locally robust survivor, which suggests that the local robustness of SEU preference only represents a knife-edge case.

This paper is organized as follows: Section 2 introduces the economy, the family of generalized recursive preferences and defines a competitive equilibrium. Section 3 characterizes the competitive equilibrium through a set of first order conditions and shows its existence. Sections 4 derives the dynamics of marginal utility ratios. Section 5 introduces our local approach to study long-run dynamics. Section 6 contains our characterization of the long-run dynamics. Section 7 discusses the robustness of the MSH. All proofs are collected in the Appendix.

2 Model Setup

In this section, we introduce the basic notions for our infinite-horizon stochastic exchange economy and define a competitive equilibrium.

2.1 Uncertainty and preferences

Time is discrete and denoted by $t \in \mathbb{T} = \{0, 1, 2, 3, ...\}$. Let S denote the finite set of states of nature. The initial state at time 0 is s_0. At each period $t \geq 1$, a state $s_t \in S$ is realized. Let Σ

denote the set of all paths with a typical element $\sigma = (s_1, s_2, ...)$. Let Σ^t denote the set of histories up to time t with a typical element $\sigma_t = (s_1, ..., s_t)$. Further define $\Sigma(\sigma_t) = \{\tilde{\sigma} \in \Sigma : \tilde{\sigma}_t = \sigma_t\}$, which denotes a cylinder set with base σ_t. Let $\mathcal{F}_t$ denote the σ-algebra that consists of all finite unions of the sets $\Sigma(\sigma_t)$, where $\mathcal{F}_0 = \{\emptyset, S\}$ is the trivial σ-algebra and $\mathcal{F}$ is the σ-algebra generated by $\cup_{t=0}^{\infty} \mathcal{F}_t$. The states of nature are drawn according to a distribution π_0 on Σ. We also use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability measure and expectation induced by π_0.

The economy is populated by I infinitely-lived consumers and there is a single good every period. With some abuse of notation, we also use I to denote the set of consumers. A consumption plan is a stochastic process $c : \Sigma \times \mathbb{T} \to \mathbb{R}_+$, where $c_t(\cdot) \equiv c(\cdot, t)$ is $\mathcal{F}_t$-measurable. The set of all consumption plans is $\mathbb{C} = L_\infty^+$, where L_∞^+ denotes the set of non-negative adapted stochastic processes on Σ that are essentially bounded. Each consumer is endowed with a consumption plan $e^i \in \mathbb{C} \backslash \{0\}$. The aggregate endowment is denoted by e. Consumers have complete preferences over all feasible consumption plans that are represented by some mapping $V : \mathbb{C} \to L_\infty^+$, where $V_{\sigma_t}(c) \equiv V(c)(\sigma, t)$ denotes the utility of consuming c at σ_t. We assume that consumers have ***recursive preferences*** over the consumption plans which take the following form

$$V_{\sigma_t}(c) = F\left(c_t, \mathcal{R}_{\sigma_t}\left(\boldsymbol{V}_{\boldsymbol{\sigma_{t+1}}}(c)\right)\right), \tag{1}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is the time aggregator which describes the attitude toward temporal resolution of uncertainty, $\mathcal{R}_{\sigma_t} : \mathbb{R}^S \to \mathbb{R}$ is the certainty equivalent aggregator which describes the attitude toward the uncertainty in the next period's utility values, and $\boldsymbol{V}_{\boldsymbol{\sigma_{t+1}}}(c) = \left(V_{\sigma_t, s}(c)\right)_{s \in S}$ is vector of utility values at date $t+1$. To simplify notation, we denote V_{σ_t} and $\mathcal{R}_{\sigma_t}$ as V_t and $\mathcal{R}_t$ henceforth. Below we introduce some examples of recursive preferences.

Example 1. (Discounted Expected Utility) Let $F(c, y) = u(c) + \beta y$, and $\mathcal{R}_t(V) = \mathbb{E}_\pi(V|\mathcal{F}_t)$. Then, the utility satisfies

$$V_t(c) = u(c_t) + \beta \times \mathbb{E}_\pi\left(V_{t+1}(c)\,|\mathcal{F}_t\right),$$

which is the discounted expected utility preference. The survival in economies with these preferences is studied by Sandroni (2000) and Blume and Easley (2006).

Example 2. (Epstein-Zin Preferences) Let $F(c, y) = \left((1-\beta)c^{1-\rho} + \beta y^{1-\rho}\right)^{\frac{1}{1-\rho}}$ and $\mathcal{R}_t(V) = \left[\mathbb{E}_\pi\left(V^{1-\gamma}(c)\,|\mathcal{F}_t\right)\right]^{\frac{1}{1-\gamma}}$. Then, the preference satisfies

$$V_t(c) = \left((1-\beta)c_t^{1-\rho} + \beta\left[\mathbb{E}_\pi\left(V_{t+1}^{1-\gamma}(c)\,|\mathcal{F}_t\right)\right]^{\frac{1-\rho}{1-\gamma}}\right)^{\frac{1}{1-\rho}},$$

which is the Epstein-Zin preference with homothetic aggregator, see Epstein and Zin (1989). Here, $1/\rho$ stands for the elasticity of intertemporal substitution (EIS), and γ stands for the coefficient of relative risk aversion. The survival in economies for particular cases of these preferences is studied by Dindo (2019) and Borovička (2020).[4]

[4] Borovička (2020)'s results are for a two-consumer economy with homogeneous preferences while Dindo (2019)'s

Example 3. (Recursive Smooth Ambiguity) Suppose that consumers face model uncertainty and entertain a set of distributions

$$\mathbb{M} = \{\pi_1, ..., \pi_M\}.$$

Let $F(c, y) = u(c) + \beta y$, and $\mathcal{R}_t(V) = \phi^{-1}(\mathbb{E}_\mu \phi[\mathbb{E}_\pi(V|\mathcal{F}_t)])$, where μ stands for the distribution over $\mathbb{M}$. So, we have

$$V_t(c) = u(c_t) + \beta \times \phi^{-1}(\mathbb{E}_\mu \phi[\mathbb{E}_\pi(V_{t+1}(c)|\mathcal{F}_t)]),$$

which is the recursive smooth ambiguity preference (Klibanoff et al., 2009). The survival in economies where consumers have these preferences and homogeneous beliefs is studied in Guerdjikova and Sciubba (2015). Condie (2008) studies the extreme case of the max-min preferences with heterogeneous beliefs.

Our setup also includes preferences that have not been studied in the previous literature on market selection. Below are two examples.

Example 4. (Kreps-Porteus Preferences) Let F be an arbitrary function that is strictly increasing in its second argument and let $\mathcal{R}_t(V) = \mathbb{E}_\pi(V|\mathcal{F}_t)$. Then, the we have the utility functions introduced by Kreps and Porteus (1978) given by

$$V_t(c) = F(c_t, \mathbb{E}_\pi(V|\mathcal{F}_t)),$$

where for some time aggregators, the concavity/convexity of F with respect to its second argument determines whether the consumer displays a preference for the late/early resolution of uncertainty.

Example 5. (Chew-Dekel Preferences) The certainty equivalent aggregator is defined implicitly as the solution to

$$\mathcal{R} = E_\pi[M(V_{t+1}, \mathcal{R})|\mathcal{F}_t],$$

where M is increasing and concave in its first argument, homogeneous of degree one and satisfies $M(k,k) = k$, see Chew (1983, 1989) and Dekel (1986). One salient example is the generalized disappointment aversion preference, see Routledge and Zin (2010), where

$$\mathcal{R}_t = \left(\sum_s \pi_s V^\alpha_{\sigma_t, s} - \theta \sum_{s: V_{\sigma_t,s} < \delta\mathcal{R}_t} \pi_s \left[(\delta\mathcal{R}_t)^\alpha - V^\alpha_{\sigma_t,s}\right]\right)^{\frac{1}{\alpha}}$$

where α is the coefficient of relative risk aversion and there is penalty that is proportional to θ on outcomes that lie below a disappointment threshold $\delta\mathcal{R}_t$. If $\theta = 0$, these preferences are SEU while if $\delta = 1$ they are the disappointment aversion preferences introduced by Gul (1991).

are obtained under the restriction that the consumers with incorrect beliefs have unit elasticity of intertemporal substitution.

In addition to previous examples, there are many other important recursive preferences that haven't been explored in the market selection literature.[5] In what follows we allow consumers to have heterogeneous preferences and use superscript i to denote consumer i's preferences.

2.2 Competitive Equilibrium

A contingent contract is a promise to deliver one unit of the consumption good contingent on the realization of the observable states of the world and zero otherwise. We assume there is a complete set of competitive markets to trade contingent contracts that open at date zero.

A price system is a stochastic process $p : \Sigma \times \mathbb{T} \to \mathbb{R}$ and we define $p(\sigma_t) \equiv p(\sigma, t)$ to be the price of a contingent contract for σ_t. Let $\mathbb{Q} = \left\{p \geq 0 : \sum_{t=0}^{\infty} \sum_{\sigma\in\Sigma^t} p\left(\sigma_t\right) < \infty\right\}$ denote the set of summable contingent commodity prices. Consumer i's problem given price system p is defined as follows:

$$
\begin{aligned}
\max_{c\in\mathbb{C}} \quad & V_0^i\left(c\right) \\
\text{s.t.} \quad & \sum_{t=0}^{\infty} \sum_{\sigma\in\Sigma^t} p\left(\sigma_t\right) \times c\left(\sigma_t\right) \leq \sum_{t=0}^{\infty} \sum_{\sigma\in\Sigma^t} p\left(\sigma_t\right) \times e^i\left(\sigma_t\right), \\
& c\left(\sigma_\tau\right) \geq 0 \text{ for all } \sigma \in \Sigma \text{ and } t \geq 1
\end{aligned}
$$

where σ_0 is given and V^i satisfies (1). A competitive equilibrium is a feasible allocation and a price system such that consumer's decisions are optimal in their budget sets.

Definition 1. A ***competitive equilibrium*** is $(c, p) \in \mathbb{C}^I \times \mathbb{Q}$ such that:

1. For every i, c^i solves consumer i's problem given price system p;
2. $\sum_i c^i\left(\sigma_t\right) = e\left(\sigma_t\right)$ for all $\sigma_t \in \Sigma_t$ and all $t \in \mathbb{T}$.

3 Characterization and Existence of Equilibrium

In this section we characterize the dynamics of competitive equilibrium consumption in terms of effective discount rates and effective beliefs and show the existence of a competitive equilibrium.

In order to characterize the competitive equilibrium using a set of equations, we introduce Assumptions 1 and 2 to ensure the time aggregator is smooth and well-behaved at the boundary and the certainty equivalent aggregator is smooth and strictly increasing in continuation utilities.

Assumption 1. *$F^i\left(x, y\right)$ is continuously differentiable, $F^i \geq 0$, and $F_1^i, F_2^i > 0$ and $\lim_{x\to 0} F_1^i\left(x, y\right) = +\infty$ for all $y \geq 0$.*

Assumption 2. *$\mathcal{R}_t^i\left(V_{t+1}\right)$ is continuously differentiable and strictly increasing in $V_{\sigma_t, s_{t+1}}^i$ for all $s_{t+1} \in S$, $\sigma_t \in \Sigma_t$, $t \geq 1$.*

[5] Examples include rank-dependent utility (Quiggin, 1982), uncertainty averse preferences (Cerreia-Vioglio et al., 2011; Strzalecki, 2013), dynamic preference for flexibility (Krishna and Sadowski, 2014), cautious expected utility (Cerreia-Vioglio et al., 2015) and dynamic mixture-averse preferences (Sarver, 2018).

The following proposition shows a competitive equilibrium can be described using a system of first-order conditions. The result is a straightforward consequence of the one-deviation property of optimal plans together with the monotonicity and boundary properties of F introduced in Assumptions 1 and 2.

Proposition 1. *Suppose Assumptions 1 and 2 hold. If (c, p) is a competitive equilibrium, then for every i and $\sigma_t \in \Sigma_t$ we must have that*

$$\frac{p(\sigma_t, s)}{p(\sigma_t)} = \frac{F_1^i\left(c^i(\sigma_t, s), \mathcal{R}_{t+1}^i\left(V_{t+2}^i\right)\right)}{F_1^i\left(c^i(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right)} \times \nabla_s \mathcal{R}_t^i\left(V_{t+1}^i\right) \times F_2^i\left(c^i(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right), \tag{2}$$

where $\nabla_s \mathcal{R}_t^i\left(V_{t+1}^i\right) \equiv \partial \mathcal{R}_t^i\left(V_{t+1}^i\right) / \partial V_{\sigma_t, s}^i$.

This result is the familiar first-order condition that the marginal rate of substitution between present and future consumption, the right hand side of (2), must be equal to their relative price at an optimal interior consumption plan. Unlike the case of discounted expected utility where the marginal rate of substitution has a simple form that depends only on consumption at σ_t and (σ_t, s_{t+1}), in this case it depends also on continuation utilities. This is because the ***generalized marginal utility*** of consumption at σ_t, $F_1^i\left(c^i(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right)$, depends not only on the consumption at σ_t but also on the certainty equivalent $\mathcal{R}_t^i$. Likewise, the marginal utility of consuming at (σ_t, s) depends not only on the consumption level at (σ_t, s) but also on the continuation utilities at dates $t+1$ and $t+2$. The latter is because the marginal utility of consuming at (σ_t, s) is equal to the marginal utility of the certainty equivalent at σ_t, F_2^i, times the marginal effect on the certainty equivalent at σ_t of the consumption at (σ_t, s), that is $\nabla_s \mathcal{R}_t^i \times F_1^i$. In what follows, we call $\nabla \mathcal{R}_t^i \equiv \left(\nabla_s \mathcal{R}_t^i\right)_{s \in S}$ the ***effective belief*** of consumer i and F_2^i the ***effective discount rate*** of consumer i. In the special case of subjective expected utility, the effective belief equals the consumer's subjective belief π^i and the effective discount rate is equal to β^i. It is worth noting that although we refer to $\nabla \mathcal{R}_t^i$ as effective beliefs, it actually represents the gradient of the certainty equivalent and may not add up to one for general preferences.

3.1 Existence of an Equilibrium

To study existence of equilibrium in this economy with countably many commodities we need to introduce some assumptions on preferences and resources.

Assumption 3. *There exists $0 < m < M < +\infty$ such that $m < e^i(\sigma_t) < M$ for all $\sigma_t \in \Sigma_t$ and $t \in \mathbb{T}$.*

Assumption 4. *$F^i(x, y)$ is concave and satisfies that $F^i(x, y) = y$ has at least one non-negative solution for every $x \geq 0$ and $F^i(0, 0) > 0$.*[6]

[6] The last part of Assumption 4 rules out the case where the minimum utility is 0 as it requires $F^i(0,0) > 0$. This requirement of Assumption 4 is for technical convenience, it could be relaxed in some interesting cases, e.g., separable time aggregator. To establish existence of equilibrium, we could allow $F^i(0,0) = 0$ if we add the assumption that F is a Blackwell contraction in y.

Assumption 5. *$\mathcal{R}_t^i(V)$ is concave and satisfies $\mathcal{R}_t^i(k \cdot \mathbf{1}) = k$ for all $k \geq 0$.*

Assumption 3 simply says that endowments are uniformly bounded. Assumptions 4 and 5 impose conditions on F and $\mathcal{R}$ that are used to prove continuity and convexity of the preference relation, a key step to show the existence of an equilibrium. Now we are ready to state our result on the existence of a competitive equilibrium in our economy.

Theorem 1. *Suppose Assumptions 1 to 5 hold, then a competitive equilibrium exists and satisfies (2).*

Our proof is based on the existence theorem by Bewley (1969) that requires preferences to be strongly monotone, weakly convex and continuous in the Mackey topology. We adapt the arguments in Marinacci and Montrucchio (2010) to show that recursive utility functions generated by a time aggregator and a certainty equivalent aggregator satisfying Assumptions 1, 2, 4 and 5 represent preferences that are continuous in the product topology.[7] Since the product topology is coarser than the Mackey topology, it follows that preferences are continuous in the Mackey topology as well. We also show that preferences are strongly monotone and weakly convex.

The assumptions in Theorem 1 allow for a wide range of preferences over consumption plans. Below are some examples of preferences that satisfy these assumptions.

Example 6. Discounted expected-utility preferences with $F(c, y) = u(c) + \beta y$ and $\mathcal{R}_t(V) = \mathbb{E}_\pi(V|\mathcal{F}_t)$, where $\beta \in (0, 1)$, and $u(0) > 0$, $u'(c) > 0$, $u''(c) < 0$ and $\lim\limits_{c \to 0} u'(c) = +\infty$, and π has full support.[8]

Example 7. Epstein-Zin preferences with an "almost" homothetic time aggregator $F(c, y) = \left((1-\beta)c^{1-\rho} + \beta y^{1-\rho}\right)^{\frac{1}{1-\rho}} + \varepsilon$ and $\mathcal{R}_t(V) = \mathbb{E}_\pi\left(V^{1-\gamma}|\mathcal{F}_t\right)^{\frac{1}{1-\gamma}}$, where $\beta, \rho \in (0, 1)$ and $\varepsilon, \gamma > 0$, and π has full support.

Example 8. Recursive smooth ambiguity preferences with

$$F(c, y) = u(c) + \beta y, \text{ and } \mathcal{R}_t(V) = \phi^{-1}\left(\mathbb{E}_\mu \phi\left[\mathbb{E}_\pi(V|\mathcal{F}_t)\right]\right),$$

where $\beta \in (0, 1)$, π has full support, u satisfy the conditions in Example 6, and ϕ is strictly increasing and satisfies $\Phi(x) \equiv -\frac{\phi'(x)}{\phi''(x)}$ is concave.[9] Below are some examples of ϕ.

- Constant relative ambiguity aversion (CRAA): $\phi(x) = a\frac{x^{1-\gamma}}{1-\gamma} + b$, where $a, \gamma > 0$.
- Constant absolute ambiguity aversion (CAAA): $\phi(x) = -a\exp(-\gamma x) + b$, where $a, \gamma > 0$.

[7] The existence and uniqueness of a utility function satisfying recursion (1) follows almost directly from Theorem 1(ii) in Marinacci and Montrucchio (2010), albeit with a slight modification. Marinacci and Montrucchio (2010) restrict attention to consumption plans c that are bounded away from 0 (i.e., $\lfloor c \rfloor_\infty > 0$ in their notation). Since we are interested in the case where $c_t \to 0$, we cannot restrict to the case in which $\lfloor c \rfloor_\infty > 0$. Instead, we assume $F^i(0, 0) > 0$ and show that their result still hold.

[8] For the discounted expected-utility case, we actually don't need $u(0) > 0$ because F is linear in y and $\beta \in (0, 1)$, and hence it is a Blackwell contraction (see footnote 6).

[9] The concavity of Φ ensures that $\mathcal{R}$ is concave (see Chapter 3 of Hardy et al. (1952)), so Assumption 5 is satisfied.

- Hyperbolic absolute ambiguity aversion (HAAA): $\phi(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma}$, where $a, \gamma > 0$.

Our characterization of the consumption dynamics relies only on equation (2) and so it extends to cases where Assumptions 4 and 5 might not hold (i.e., when F is convex in its second argument) provided an equilibrium still exists.

4 Consumption Dynamics

In this section, we introduce the survival notions and derive a recursive description of consumption dynamics in the same spirit of Sandroni (2000) and Blume and Easley (2006).

Definition 2. We say consumer i ***survives*** on a path σ if $\limsup_{t\to\infty} c_t^i(\sigma) > 0$, ***vanishes*** on a path σ if $\lim_{t\to\infty} c_t^i(\sigma) = 0$ and ***dominates*** on a path σ if all other consumers vanish on that path.

In what follows, we are interested in consumers' survival in the competitive equilibrium, and we focus on the equilibrium consumption allocations without explicitly mentioning them. We analyze consumption dynamics by examining the dynamics of the marginal utility ratio between any two consumers. Lemma 1 introduces a key property employed throughout this paper.

Lemma 1. *Under Assumptions 1-3 and $\mathcal{R}_t(k\cdot\mathbf{1}) = k$ for all $k \geq 0$, we have*

$$L^{ij}(\sigma_t) \equiv \frac{F_1^i\left(c^i(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right)}{F_1^j\left(c^j(\sigma_t), \mathcal{R}_t^j\left(V_{t+1}^j\right)\right)} \to \infty \Rightarrow c^i(\sigma_t) \to 0 \quad \mathbb{P}\text{-a.s.},$$

*where $L^{ij}(\sigma_t)$ is the **generalized marginal utility ratio** between i and j.*[10]

Lemma 1 says that consumer i vanishes on all paths where the generalized marginal utility ratio between i and any other consumer j goes to infinity.

To study the dynamics of marginal utility ratios, we divide the first-order conditions (2) of any pair of consumers i and j to obtain the following equation

$$L^{ij}(\sigma_t, s_{t+1}) = L^{ij}(\sigma_t) \times B^{ji}(\sigma_t, s_{t+1}) \times D^{ji}(\sigma_t), \tag{3}$$

which provides a recursive description of the marginal utility ratio between i and j, where

$$B^{ji}(\sigma_t, s) = \frac{\nabla_s \mathcal{R}_t^j\left(V_{t+1}^j\right)}{\nabla_s \mathcal{R}_t^i\left(V_{t+1}^i\right)}, \text{ and } D^{ji}(\sigma_t) = \frac{F_2^j\left(c^j(\sigma_t), \mathcal{R}_t^j\left(V_{t+1}^j\right)\right)}{F_2^i\left(c^i(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right)}, \tag{4}$$

are the ***effective belief ratio*** and ***discount rate ratio*** respectively between j and i. From (3), we see that if i's effective belief on state s and effective discount rate are higher than j's (i.e., $B^{ji}(\sigma_t, s)$ and $D^{ji}(\sigma_t)$ are greater than 1), then i's generalized marginal utility decreases relative to j when the next-period state realization is s. Intuitively, if i has a higher effective belief on state

[10] We actually only need the weaker condition that the process $\mathcal{R}(k) \equiv \{\mathcal{R}_t(k\cdot\mathbf{1})\}_{t\in\mathbb{T}} \in L_\infty$ for each $k \geq 0$.

s, it means that the marginal effect of next-period utility in state s on the certainty equivalent is higher, so i tends to allocate more consumption to state s; also, if i has a higher effective discount rate, then it means that i is effectively more patient and tends to allocate more consumption to the next period. Both effects contribute to the decrease of generalized marginal utility ratio between i and j. Below are some special cases of the dynamics.

Example 9. (Dynamics of discounted EU) Suppose that consumers have discounted expected utility preferences with i.i.d. beliefs. Then, (3) becomes

$$\frac{u_i'\left(c^i\left(\sigma_{t+1}\right)\right)}{u_j'\left(c^j\left(\sigma_{t+1}\right)\right)} = \frac{u_i'\left(c^i\left(\sigma_{t}\right)\right)}{u_j'\left(c^j\left(\sigma_{t}\right)\right)} \times \frac{\pi^j\left(s_{t+1}\right)}{\pi^i\left(s_{t+1}\right)} \times \frac{\beta^j}{\beta^i}, \tag{5}$$

which corresponds to the dynamics in Sandroni (2000), Blume and Easley (2006, 2009). Suppose that the state distribution is also i.i.d., then (5) implies that

$$\begin{aligned}\lim_{T\to+\infty} \frac{1}{T} \log \frac{u_i'\left(c^i\left(\sigma_T\right)\right)}{u_j'\left(c^j\left(\sigma_T\right)\right)} &= \lim_{T\to+\infty} \frac{1}{T} \sum_{t=1}^{T} \log \frac{\pi^j\left(s_t\right)}{\pi^i\left(s_t\right)} + \log \frac{\beta^j}{\beta^i} \\ &\to I\left(\pi^i\right) - I\left(\pi^j\right) + \log \frac{\beta^j}{\beta^i} \quad \mathbb{P}-a.s.,\end{aligned}$$

where $I\left(\pi^i\right) = -\mathbb{E}\log\left[\pi^i\left(s\right)\right]$ represents the relative entropy of π^i. Suppose that $\beta^i = \beta^j$, then the consumer with a larger entropy—thus less precise belief—will vanish in the limit, as predicted by the market selection hypothesis.

In Example 9, marginal utility ratio can be characterized by a simple Markov process, where the transition function only depends on beliefs and discount rates. This property allows us to conveniently investigate the dynamics of marginal utility ratio by analyzing a stochastic process which is *exogenous* to the economic system without having to solve for the equilibrium. Notice that this property relies on that the discounted expected-utility preference has a linear certainty equivalent aggregator and separable time aggregator. The linearity implies that V_{t+1} does not enter in the first-order condition, so the effective beliefs and discount rates are independent of the continuation utility. However, with general recursive preferences, today's utility can depend on the continuation utility in a non-linear way, so effective belief and discount rate can depend on continuation utility and are *endogenously* determined in equilibrium. Below are some examples.

Example 10. (Dynamics of Epstein-Zin) Suppose that consumers have Epstein-Zin preferences and i.i.d. beliefs. Then, effective discount rates and beliefs are

$$F_2^i = \underbrace{\left(\frac{V_{\sigma_t}^i}{\mathcal{R}_t^i(V_{t+1}^i)}\right)^{\rho_i}}_{\text{time adjustment}} \beta^i, \quad \nabla_s \mathcal{R}_t^i\left(V_{t+1}^i\right) = \underbrace{\left(\frac{V_{\sigma_t,s}^i}{\mathcal{R}_t^i(V_{t+1}^i)}\right)^{-\gamma_i}}_{\text{risk adjustment}} \pi^i\left(s\right), \tag{6}$$

and the generalized marginal utility is $F_1^i = \left(V_{\sigma_t}^i / c_t^i\right)^{\rho_i}$. The dynamics of generalized marginal utility ratios are described by

$$L^{ij}(\sigma_t, s_{t+1}) = L^{ij}(\sigma_t) \times \frac{\left(\frac{V^j_{\sigma_t, s_{t+1}}}{\mathcal{R}^j_t(V^j_{t+1})}\right)^{-\gamma_j} \pi^j(s_{t+1})}{\left(\frac{V^i_{\sigma_t, s_{t+1}}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{-\gamma_i} \pi^i(s_{t+1})} \times \frac{\left(\frac{V^j_{\sigma_t}}{\mathcal{R}^j_t(V^j_{t+1})}\right)^{\rho_j} \beta^j}{\left(\frac{V^i_{\sigma_t}}{\mathcal{R}^i_t(V^i_{t+1})}\right)^{\rho_i} \beta^i}. \tag{7}$$

Compared with the SEU case (5), we have time and risk adjustments in front of the standard beliefs and discount rates. The time adjustment depends on the current utility, $V^i_{\sigma_t}$, relative to the date t certainty equivalent of future utility: When current utility is relative low (high), the consumer becomes more impatient (patient) and discounts the future more (less) heavily. The risk adjustment of state s_{t+1} depends on the future utility in that state, $V^i_{\sigma_t, s_{t+1}}$, relative to the the date t certainty equivalent of future utility: States where utility is relatively low (high) are overweighted (underweighted).

Example 11. (Dynamics of smooth ambiguity) Suppose that consumers have recursive smooth ambiguity preferences and i.i.d. beliefs. Then, effective discount rates and beliefs become

$$F^i_2 = \beta^i, \quad \nabla_s \mathcal{R}^i_t\left(V^i_{t+1}\right) = \mathbb{E}_{\mu^i}\left(\underbrace{\frac{\phi'_i\left[\mathbb{E}_\pi\left(V^i_{t+1}(c)\right)\right]}{\phi'_i\left(\mathcal{R}^i_t(V^i_{t+1})\right)}}_{\text{ambiguity adjustment}} \pi(s)\right). \tag{8}$$

and the generalized marginal utility is $F^i_1 = u'(c_i)$. Note that effective discount rates and the generalized marginal utility are both standard because F is separable. The dynamics of generalized marginal utility ratios follow

$$L^{ij}(\sigma_t, s_{t+1}) = L^{ij}(\sigma_t) \times \frac{\mathbb{E}_{\mu^j}\left(\frac{\phi'_j\left[\mathbb{E}_\pi\left(V^j_{t+1}(c)\right)\right]}{\phi'_j\left(\mathcal{R}^j_t(V^j_{t+1})\right)} \times \pi(s_{t+1})\right)}{\mathbb{E}_{\mu^i}\left(\frac{\phi'_i\left[\mathbb{E}_\pi\left(V^i_{t+1}(c)\right)\right]}{\phi'_i\left(\mathcal{R}^i_t(V^i_{t+1})\right)} \times \pi(s_{t+1})\right)} \times \frac{\beta^j}{\beta^i}, \tag{9}$$

which corresponds to the dynamics in Guerdjikova and Sciubba (2015). When consumers are ambiguity neutral, ϕ' is constant and (9) becomes (5) with $\pi^i = \mathbb{E}_{\mu^i}(\pi)$. When consumers are ambiguity-averse, there is ambiguity adjustment for each model. The ambiguity adjustment of model π depends on the expected future utility evaluated under π relative to the date t certainty equivalent of future utility: Models where the expected future utility is relatively low (high) are overweighted (underweighted).

Example 12. (Dynamics of Generalized Disappointment Aversion) Suppose that consumers have generalized disappointment aversion preferences with i.i.d. beliefs and a separable time aggregator.

Then, effective discount rates and beliefs are[11]

$$F_2^i = \beta^i, \quad \nabla_s \mathcal{R}_t^i \left(V_{t+1}^i\right) = \underbrace{\alpha \left(\frac{V_{\sigma_t,s}^i}{\mathcal{R}_t^i}\right)^{\alpha-1} \left(\frac{1+\theta^i \cdot \mathbf{1}_{\{V_{\sigma_t,s}^i < \delta^i \mathcal{R}_t^i\}}}{1+(\delta^i)^\alpha \theta^i \cdot \sum_{s'} \pi^i(s') \cdot \mathbf{1}_{\{V_{\sigma_t,s'}^i < \delta^i \mathcal{R}_t^i\}}}\right)}_{\text{disappointment adjustment}} \pi^i(s), \quad (10)$$

and the generalized marginal utility is $F_1^i = u'(c_i)$. The dynamics of generalized marginal utility ratios are described by

$$L^{ij}\left(\sigma_t, s_{t+1}\right) = L^{ij}\left(\sigma_t\right) \times \frac{\alpha_j \left(\frac{V_{\sigma_t,s_{t+1}}^j}{\mathcal{R}_t^j}\right)^{\alpha_j-1} \left(\frac{1+\theta^j \cdot \mathbf{1}_{\{V_{\sigma_t,s_{t+1}}^j < \delta^j \mathcal{R}_t^j\}}}{1+(\delta^j)^\alpha \theta^j \cdot \sum_s \pi^j(s) \cdot \mathbf{1}_{\{V_{\sigma_t,s}^j < \delta^j \mathcal{R}_t^j\}}}\right) \pi^j(s_{t+1})}{\alpha_i \left(\frac{V_{\sigma_t,s_{t+1}}^i}{\mathcal{R}_t^i}\right)^{\alpha_i-1} \left(\frac{1+\theta^i \cdot \mathbf{1}_{\{V_{\sigma_t,s}^i < \delta^i \mathcal{R}_t^i\}}}{1+(\delta^i)^\alpha \theta^i \cdot \sum_s \pi^i(s) \cdot \mathbf{1}_{\{V_{\sigma_t,s}^i < \delta^i \mathcal{R}_t^i\}}}\right) \pi^i(s_{t+1})} \times \frac{\beta^j}{\beta^i}. \quad (11)$$

Compared with the SEU case (7), we have disappointment adjustments in front of standard beliefs. In the case $\alpha = 1$, the disappointment adjustment overweights all the disappointing events by the same factor and underweights the probabilities of the complementary events. In the case $\alpha < 1$, the disappointment adjustment involves an additional term, $\left(\frac{V_{\sigma_t,s_{t+1}}^i}{\mathcal{R}_t^i}\right)^{\alpha_i-1}$, that is a decreasing functon of the ratio between future utility in that state, $V_{\sigma_t,s_{t+1}}^i$, relative to the the date t certainty equivalent of future utility. Consequently, compared to the case $\alpha = 1$, disappointing events receive an additional weight that is larger the more disappointing the event is.

From previous examples, we see that the evolution of the generalized marginal utility can be complicated as both effective beliefs and discount rates may be endogenous. To characterize the consumption dynamics, this paper adopts a "local approach" by studying the approximated dynamics when a specific consumer almost dominates the market, i.e, consumes almost all the endowment. Suppose that consumer i dominates on a path. Then, it turns out that when t is sufficiently large, c_t^i can be approximated by the total endowment, and V_{t+1}^i can be approximated by be the utility of consuming the aggregate endowment forever, and correspondingly, the consumption of all other consumers $j \neq i$ is approximately 0, and their utility V_{t+1}^j approaches the utility of consuming zero forever. Therefore, we can approximate endogenous terms in (6), e.g., V_{t+1}, by terms that are exogeneously determined. This approach will be explained in greater detail in the following sections.

5 Local Survival Index

In this section we introduce some concepts to implement the aforementioned local approach. For simplicity, we focus on the stationary case and impose the following assumption.

[11] Although these preferences are not differentiable at the point $V_{\sigma_t,s} = \delta \mathcal{R}_t$, we can approximate them by a smooth version for which the above FOC holds as noted by Routledge and Zin (2010).

Assumption 6. *(Stationarity) States and endowments are i.i.d, and $\mathcal{R}_t^i = \mathcal{R}^i$ for all $t \in \mathbb{T}$.*

Assumption 6 posits a stationary structure on uncertainty. First, uncertainty manifests itself in a time-independent manner, i.e., state and endowment are i.i.d. distributed. Second, consumers have time-independent preference toward uncertainty, e.g., consumers hold i.i.d. beliefs and there is no learning. This assumption simplifies the analysis and allows us to focus on the pure effect of market selection.[12]

Definition 3. (Local utility) Under Assumption 6, we define $\underline{V}^i = V_{\sigma_t}^i(0)$ and $\overline{V}_{s_t}^i = V_{\sigma_t}^i(e)$ for all $s_t \in S$ and $i \in I$.

In words, $\underline{V}^i$ denotes the continuation utility of consumer i when she consumes 0 in every period, and $\overline{V}_{s_t}^i$ denotes the continuation utility of i when the current state is s_t and she consumes the total endowment e in every period. Under Assumption 6, $\underline{V}^i$ is fixed and $\overline{V}_{s_t}^i$ only depends on the current state s_t, and they can be simply obtained by solving a system of equations.[13] We then define the local survival index as follows.

Definition 4. (Local survival index) For all i, j, we define

$$\mathcal{S}_i^j = \begin{cases} \mathbb{E}\log\left[\nabla_s \mathcal{R}^i\left(\overline{V}^i\right)\right] + \mathbb{E}\log F_2^i\left(e(s), \mathcal{R}^i\left(\overline{V}^i\right)\right), & \text{if } j = i \\ \mathbb{E}\log\left[\nabla_s \mathcal{R}^j\left(\underline{V}^j\right)\right] + \mathbb{E}\log F_2^j\left(0, \mathcal{R}^j\left(\underline{V}^j\right)\right), & \text{if } j \neq i. \end{cases}$$

called the ***local survival index*** of consumer j when i dominates.

The local survival index $\mathcal{S}_i^j$ has two components. The first component is the expected log, or the negative of the relative entropy, of consumer j's effective belief when i dominates.[14] The second component is the expected log of consumer j's effective discount rate when i dominates. A consumer has a higher survival index if her effective belief has a lower entropy, and if her effective discount rate is higher. A special case is the discounted expected utility where $\mathcal{S}_i^j = -I(\pi^j) + \log\beta^j$ for all i, j. We then introduce the following concept.

Definition 5. (Local dominance) For all i, j, we denote by

$$i \succ_i j \quad \text{if } \mathcal{S}_i^i > \mathcal{S}_i^j,$$

and we say that i ***locally dominates*** j. We denote by $j \succ_i i$ if the inequality is reversed.

[12]We believe that our approach can extend to situations with time heterogeneity by appropriately modifying notations. For example, suppose there is learning but beliefs converge to some limit, then the exact same analysis applies here; also, we can re-define the entropy using its time average to characterize dynamics for non-i.i.d. cases.

[13]From the definition of recursive utility (1), $\underline{V}_s^i$ and $\overline{V}_s^i$ must satisfy

$$\underline{V}^i = F^i\left(0, \mathcal{R}^i\left(\underline{V}^i\right)\right) \text{ and } \overline{V}_s^i = F^i\left(e(s), \mathcal{R}^i\left(\overline{\boldsymbol{V}}^i\right)\right) \text{ for all } s \in S,$$

where $\overline{\boldsymbol{V}}^i = \left(\overline{V}_s^i\right)_{s\in S}$ is the vector of the maximum local utility.

[14]With some abuse of language, we refer to $-\mathbb{E}\log\left(\nabla_s\mathcal{R}^i\right)$ as the relative entropy of i's effective belief though $\nabla\mathcal{R}^i$ may not be a probability.

Definition 5 says that i locally dominates j if she has a higher survival index than j when she dominates the market, i.e., consumes the total endowment. The local dominance is closely related to the consumption dynamics. Intuitively, when i consumes almost everything, dynamics (3) can be approximated by

$$L^{ij}\left(\sigma_{t+1}\right) \approx L^{ij}\left(\sigma_{t}\right) \times \frac{\nabla_{s} \mathcal{R}^{j}\left(\underline{V}^{j}\right)}{\nabla_{s} \mathcal{R}^{i}\left(\overline{V}^{i}\right)} \times \frac{F_{2}^{j}\left(0, \mathcal{R}^{j}\left(\underline{V}^{j}\right)\right)}{F_{2}^{i}\left(e\left(s\right), \mathcal{R}^{i}\left(\overline{V}^{i}\right)\right)}, \tag{12}$$

where each consumer's utility is approximated by the local utility when i dominates.[15] (12) implies that

$$\mathbb{E} \log \frac{L^{ij}\left(\sigma_{t+1}\right)}{L^{ij}\left(\sigma_{t}\right)} \approx \mathcal{S}_{i}^{j} - \mathcal{S}_{i}^{i}.$$

Therefore, $i \succ_i j$ implies that the log marginal utility ratio between i and j decreases on expectation when i consumes almost everything, which further implies that i's consumption has a tendency to increase relative to j's when i consumes almost everything. With expected utility, each consumer j has a fixed survival index $\mathcal{S}_{i}^{j} = \mathcal{S}^{j}$ across all i, so the relative consumption dynamics remain identical when a consumer dominates the market and when she vanishes. This leads to a complete ranking among all consumers based on their survival index, where only those with the highest index can survive (Sandroni (2000), Blume and Easley (2006)). However, with general utilities, the survival index can vary with the consumption allocation and may result in reversed ranking, such as $i \succ_i j$ and $j \succ_j i$ both holding true. This suggests that the rankings based on survival index can be incomplete, leading to a variety of survival patterns different from those in the standard case.

6 Characterizations of Dynamics

This section presents our characterization of the consumption dynamics using the aforementioned local approach. To implement the local approach, we introduce an irreducibility assumption, that is every consumer has positive probability of almost dominating the market, or equivalently, the marginal utility ratio between any consumer i and all other consumers can be arbitrarily small with positive conditional probability at every date.

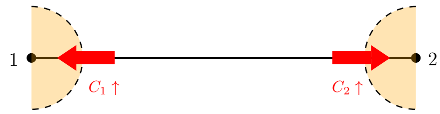
Assumption 7. *(Irreducibility) For all $0 < \underline{L} < \overline{L} < +\infty$ and $i \in I$, there exist $K < \infty$ such that whenever $\max_{j \in I} L^{ij}\left(\sigma_{t}\right) < \overline{L}$, we have*

$$\max_{j \in I} L^{ij}\left(\sigma_{t}, s_{1}, ..., s_{K}\right) < \underline{L} \text{ for some } s_{1}, ..., s_{K} \in S,$$

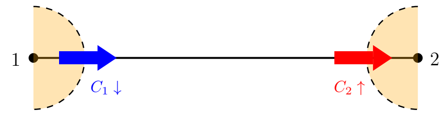
for all $t \in \mathbb{T}$ and $\sigma_{t} \in \Sigma_{t}$.

Notice this is an assumption on endogenous variables as L^{ij} depends on the equilibrium consumption allocation. While in the Appendix A.5 we provide conditions on exogenous variables to ensure this

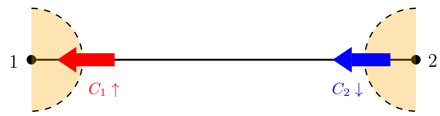
[15] This approximation arises from the continuity of preferences in the product topology, which will elaborated in the proof sketch of Theorem 2 in Section 6.



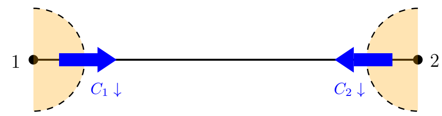
(a): $1 \succ_1 2$ and $2 \succ_2 1$



(b): $2 \succ_1 1$ and $2 \succ_2 1$



(c): $1 \succ_1 2$ and $1 \succ_2 2$



(d): $2 \succ_1 1$ and $1 \succ_2 2$

Figure 1: Two-consumer case

assumption holds,[16] we retain Assumption 7 in the main text because it explicitly highlights the crucial property of the marginal utility ratio used in the proof. From a technical standpoint, we consider it a natural assumption because even when it fails for certain consumers, we can still apply our characterization to those consumers for whom it holds (which is why we call it 'irreducibility'). Moreover, we demonstrate that violations of Assumption 7 are not robust to small perturbations, as explained in more detail in Section 7.1. Therefore, this assumption is convenient for the purpose of exploring the robustness of the MSH.

6.1 Two-consumer case

We first present our characterization for the two-consumer case.

Theorem 2. *(Two-consumer) Suppose that $|I| = 2$. Under Assumptions 1 to 7, we have:*

(i) if $i \succ_i j$ and $i \succ_j j$, then i survives and j vanishes $\mathbb{P}$-almost surely;

(ii) if $i \succ_j j$ and $j \succ_i i$, then i and j survive together $\mathbb{P}$-almost surely;

(iii) if $i \succ_i j$ and $j \succ_j i$, then only one consumer survives $\mathbb{P}$-almost surely, and each survives with a $\mathbb{P}$-strictly positive probability.

Theorem 2 shows that a variety of patterns can emerge with non-expected utility unlike the SEU case when there is always a unique survivor except for tie cases. Figure 1 depicts the four possible cases of the local dominance relation. Figure 1(a) illustrates Theorem 2(iii) in which both consumer 1 and 2 are locally dominant. To see that, note that $1 \succ_1 2$ implies that when consumer 1 almost dominates the market, her consumption increases relative to that of consumer 2 in expectation. This relation reinforces the initial dominating position of consumer 1, so she will eventually dominate the market with a strictly positive probability, and the argument is symmetric for consumer 2. Assumption 7 further guarantees that each neighborhood where some consumer almost dominates will be visited with a uniformly positive conditional probability, which establishes

[16] For example, the assumption holds as long as consumers' disagreement, as captured by their belief ratio, about one of the states is large enough.

a global result that both consumers can dominate regardless of the initial condition, and one of them dominates almost surely. Other cases can be discussed similarly.

Proof sketch of Theorem 2

Here, we focus on explaining why $i \succ_i j$ implies consumer i dominates the market with a strictly positive probability. The claim proceeds in the following steps.

Step 1: We prove that every consumer's utility can be approximated by their local utility (Definition 3) when the marginal utility ratio is sufficiently small (or large). More precisely, we show that for all $t \in \mathbb{T}$, $\sigma \in \Sigma$ and $\varepsilon > 0$, there exists $\underline{L} > 0$ such that

$$L^{ij}(\sigma_t) < \underline{L} \text{ implies that } |V^i(\sigma_t) - \overline{V}^i| < \varepsilon \text{ and } |V^j(\sigma_t) - \underline{V}^j| < \varepsilon. \tag{13}$$

This is because under our assumptions, effective belief and discount rate ratios are bounded, so when the current marginal utility ratio L_t^{ij} is sufficiently small, the future marginal utility ratios $L_{t+1}^{ij}, ..., L_{t+K}^{ij}$ for any $K < \infty$ can be arbitrarily small, which implies that $c_t^j, ..., c_{t+K}^j$ can be arbitrarily close to 0. Lemma 5 in the Appendix shows that V_{σ_t} is continuous in product topology, so sufficiently many (but still finite) periods of low consumption imply that the utility is close to the utility of consuming 0 forever, which establishes (13).

Step 2: We construct a modified process of marginal utility ratio between i and j and show that its log is a local supermartingale when i almost dominates the market. Specifically, we construct a process $\left\{\hat{L}^{ij}\right\}$ where

$$\hat{L}^{ij}(\sigma_t, s_{t+1}) = \hat{L}^{ij}(\sigma_t) \times B^{ji}(\sigma_t, s_{t+1}) \times D^{ji}(\sigma_t, s_{t+1}). \tag{14}$$

Note that only change from (3) is that we shift forward the ratio of discount rate by one period. This modification allows us to construct a local supermartingale with increment given by the difference of local survival index. That is, we have

$$\mathbb{E}\left(\log\left(\hat{L}_{t+1}^{ij}\right)|\mathcal{F}_t\right) \approx \log\left(\hat{L}_t^{ij}\right) + \mathcal{S}_i^j - \mathcal{S}_i^i < \log\left(\hat{L}_t^{ij}\right),$$

when L_t^{ij} is sufficiently small. We then show that the local supermartingale property further implies that $\hat{L}_t^{ij} \to 0$ with a strictly positive probability. We note that the effective discount rate ratios are bounded, so the ratio between $\hat{L}_t^{ij}$ and L_t^{ij} is also bounded, so $\hat{L}_t^{ij} \to 0$ implies $L_t^{ij} \to 0$, which further implies that consumer i dominates the market with positive probability.

Similarly, we can prove that $j \succ_i i$ implies that consumer i dominates the market with 0 probability. The irreducibility assumption implies that every neighborhood is reachable, so

Theorem 2 follows from different combinations of the local dominance relation.

Examples

Below we illustrate Theorem 2 using some examples of non-expected utility preferences that generate survival patterns absent in the expected-utility framework.

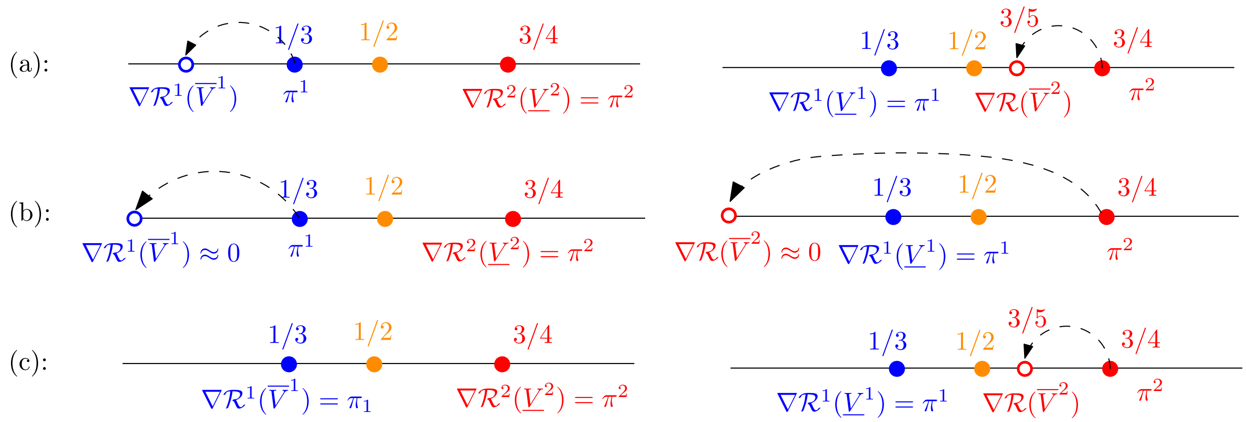


Figure 2: Survival with non-expected utility in Example 13

Note: All beliefs depicted in the figure, e.g., π_i and $\nabla\mathcal{R}^i$, denote the probability on the first state, s_1. Graphs on the left (right) depict effective beliefs when consumer 1 (2) consumes all the endowment.

Example 13. Suppose that $S = \{s_1, s_2\}$, $I = \{1, 2\}$. Consumers have i.i.d. subjective beliefs $\pi^1 = (1/3, 2/3)$, $\pi^2 = (3/4, 1/4)$, and the true state distribution is $\pi_0 = (1/2, 1/2)$. The total endowment in each state is expressed as utilities with $u(s_1) = 2$ and $u(s_2) = 1$, where $u(s) \equiv u(e(s))$ denotes the utility of consuming the aggregate endowment in state s. Consumers have the following preferences

$$V^i_{\sigma_t} = u(c_t) - \beta \times \frac{1}{\gamma_i} \log \left(\sum_{s_{t+1} \in S} \exp\left(-\gamma_i V^i_{\sigma_t, s_{t+1}}\right) \pi^i(s_{t+1}) \right),$$

where $\gamma_i > 0$ is the degree of ambiguity aversion.[17] When i consumes all the endowment, both consumers' effective beliefs are

$$\nabla_s \mathcal{R}^j\left(\underline{V}^j\right) = \pi^j(s) \text{ and } \nabla_s \mathcal{R}^i\left(\overline{V}^i\right) = \frac{\exp(-\gamma_i u(s)) \pi^i(s)}{\sum_{s' \in S} \exp(-\gamma_i u(s')) \pi^i(s')}. \tag{15}$$

We see that i's effective belief overweights the bad state, but j's effective belief is unadjusted because she consumes 0 in every state and so bears no uncertainty. The case where j consumes all

[17] This is a special case of the smooth ambiguity preference where the first-order distributions are Dirac beliefs. This formulation is to simplify the exposition so that we don't need to discuss beliefs over models, and then over states, but the exact same analysis extends to general smooth ambiguity preferences.

the endowment is symmetric. First notice that, consumer 1's subjective belief is more precise than consumer 2's in the sense that it has a lower relative entropy. As illustrated in Figure 2, $\pi_1 = 1/3$ is closer to the true probability $\pi_0 = 1/2$ than $\pi_2 = 3/4$. From Sandroni (2000) and Blume and Easley (2006), we know that consumer 1 will drive out consumer 2 if they have SEU preferences. However, with non-expected utility preferences, we can have the following patterns.

1. *The less correct consumer dominates.* Suppose that both consumers are ambiguity-averse with $\gamma_1 = \gamma_2 = \ln 2$, then

$$\begin{cases} \nabla \mathcal{R}^1 \left(\overline{V}^1\right) & = (1/5, 4/5) \\ \nabla \mathcal{R}^2 \left(\underline{V}^2\right) & = (3/4, 1/4) \end{cases} \Rightarrow \mathcal{S}_1^1 < \mathcal{S}_1^2, \text{and } \begin{cases} \nabla \mathcal{R}^1 \left(\underline{V}^1\right) & = (1/3, 2/3) \\ \nabla \mathcal{R}^2 \left(\overline{V}^2\right) & = (3/5, 2/5) \end{cases} \Rightarrow \mathcal{S}_2^1 < \mathcal{S}_2^2,$$

so we have $2 \succ_1 1$ and $2 \succ_2 1$, and consequently, consumer 2 dominates the market. This is illustrated in Figure 2(a), where the first and second graphs depict the effective beliefs when consumers 1 and 2 consume the aggregate endowment, respectively. First note that s_2 is the worse state because it has a lower aggregate endowment. Therefore, both consumers' effective beliefs will overweight s_2—and hence underweight s_1—when they consume the aggregate endowment. In Figure 2(a), this means that $\nabla \mathcal{R}^i(\overline{V}^i)$ moves to the left of π_i for both $i = 1, 2$. Let's first consider the left graph in which consumer 1 consumes all endowment. Notice that consumer 1 underestimates the probability of s_1, which, in the graph, means that π_1 is on the left of $1/2$. Therefore, consuming all the endowment amplifies her underestimation by shifting her effective belief further to the left. This makes her effective belief less correct than π_2, and we have $2 \succ_1 1$. Now, let's consider the right graph in which consumer 2 consumes all endowment. In contrast, consumer 2 overestimates the probability of s_1, meaning that π_2 is on the right of $1/2$. When consumer 2 consumes all the endowment, her effective belief on s_1 also shifts to the left, but this belief adjustment compensates for her initial belief overestimation. This makes her effective belief more correct than consumer 1's, and we have $2 \succ_2 1$. Since consumer 2 locally dominates consumer 1 in both neighborhoods, she will dominate the market almost surely as stated in Theorem 2 (i).

2. *Co-existence.* Suppose that γ_1, γ_2 sufficiently large, say $\gamma_1, \gamma_2 \to +\infty$. Then, we have

$$\begin{cases} \nabla \mathcal{R}^1 \left(\overline{V}^1\right) & \to (0, 1) \\ \nabla \mathcal{R}^2 \left(\underline{V}^2\right) & = (3/4, 1/4) \end{cases} \Rightarrow \mathcal{S}_1^1 < \mathcal{S}_1^2, \text{and } \begin{cases} \nabla \mathcal{R}^1 \left(\underline{V}^1\right) & = (1/3, 2/3) \\ \nabla \mathcal{R}^2 \left(\overline{V}^2\right) & \to (0, 1) \end{cases} \Rightarrow \mathcal{S}_2^1 > \mathcal{S}_2^2,$$

so we have $2 \succ_1 1$ and $1 \succ_2 2$, which corresponds to the co-existence of both consumers, as illustrated in Figure 2 (b). Let's first consider the left graph, in which consumer 1 consumes all endowment. In this case, consumer 1 bears the aggregate uncertainty, and when she is extremely ambiguity-averse, her effective belief assigns almost all weight to the worse state, s_2. In contrast, consumer 2 bears no uncertainty, and her effective belief is undistorted and equal to her subjective belief. Therefore, consumer 1's effective belief is less accurate than consumer 2's, resulting in $2 \succ_1 1$. The case in the right graph, where consumer 2 consumes all the endowment, is symmetric.

In this case, consumer 2's effective belief approaches a point-mass belief on s_2, leading to $1 \succ_1 2$. Since both consumers are locally dominated by each other, they co-exist in the market as indicated by Theorem 2 (ii).

3. *Each consumer dominates with a positive probability.* Suppose $\gamma_1 = 0$ and $\gamma_2 = \ln 2$, then

$$\begin{cases} \nabla \mathcal{R}^1 \left(\overline{V}^1\right) & = (1/3, 2/3) \\ \nabla \mathcal{R}^2 \left(\underline{V}^2\right) & = (3/4, 1/4) \end{cases} \Rightarrow \mathcal{S}_1^1 > \mathcal{S}_1^2, \text{and} \begin{cases} \nabla \mathcal{R}^1 \left(\underline{V}^1\right) & = (1/3, 2/3) \\ \nabla \mathcal{R}^2 \left(\overline{V}^2\right) & = (3/5, 2/5) \end{cases} \Rightarrow \mathcal{S}_2^1 < \mathcal{S}_2^2,$$

so we have $1 \succ_1 2$ and $2 \succ_2 1$, and hence each consumer dominates with a strictly positive probability. This case is depicted in Figure 2(c). Since consumer 1 has standard SEU preferences, her effective belief is always equal to her subjective belief. In the left graph where consumer 1 consumes all the endowment, both consumers' effective beliefs are their subjective beliefs, leading to $1 \succ_1 2$ because consumer 1 has a more precise belief. In the right graph where consumer 2 consumes all the endowment, consumer 2's effective belief shifts to the left, bringing it closer to $1/2$ than π_1. Consequently, we have $2 \succ_1 1$. Since both consumers locally dominate each other, both can dominate with a strictly positive probability as indicated by Theorem 2 (iii).

The preferences we assumed in Example 2 imply that the effective beliefs when some consumer dominates the market are probability measures, so a consumer with correct beliefs can't be driven out of the market by consumers with incorrect beliefs. The following example shows that for some other preferences a consumer i with incorrect beliefs can actually drive another with correct beliefs out of the market almost surely if the mass i allocates to future states becomes larger than one.

Example 14. Suppose $I = \{1, 2\}$, $S = \{s_1, s_2\}$ and the true state distribution consists of i.i.d. draws with $\pi_0 = \left(\frac{1}{2}, \frac{1}{2}\right)$. Consumers have i.i.d. subjective beliefs $\pi^1 = \left(\frac{3}{4}, \frac{1}{4}\right)$, $\pi^2 = \pi_0$ and preferences given by

$$V_t^i(c) = u(c_t) + \beta \left[\mathbb{E}_{\pi^i}\left(\left(V_{t+1}^i\right)^{1-\gamma_i}(c) \,|\mathcal{F}_t\right)\right]^{\frac{1}{1-\gamma_i}}$$

where $u(0) > 0$. Effective discount rates are identical and given by $F_2^i = \beta$, thus irrelevant for survival. When i dominates the market, the consumers' effective beliefs are

$$\nabla_s \mathcal{R}^j \left(\underline{V}^j\right) = \pi^j(s) \text{ and } \nabla_s \mathcal{R}^i \left(\overline{V}^i\right) = \left(\frac{\overline{V}_s^i}{\mathcal{R}^i(\overline{V}^i)}\right)^{-\gamma_i} \pi^i(s), \tag{16}$$

Note that the effective beliefs of the consumer who consumes all the endowment need not add up to one. We have:

The correct consumer vanishes. Let $\gamma_1 = \gamma_2 = \frac{3}{2}$. Let $\overline{V}_1^1 = 4$, $\overline{V}_2^1 = 1$ and $\mathcal{R}^1(\overline{V}^1) = \frac{64}{25}$.[18]

[18] These values can be obtained by letting $\beta = \frac{25}{128}$, $u(e(1)) = 3.5$ and $u(e(2)) = 0.5$.

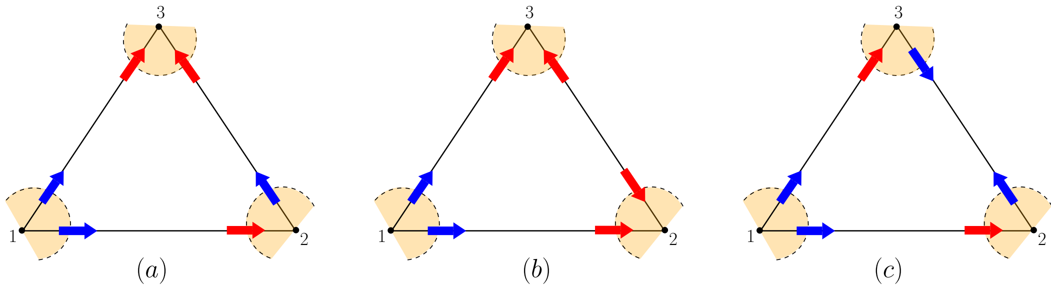


Figure 3: Multiple Consumers

Then, $\overline{V}_1^2 \approx 3.791$, $\overline{V}_2^2 \approx 0.791$ and $\mathcal{R}^2(\overline{V}^2) \approx 1.491$. It is easy to show that:

$$\begin{cases} \nabla \mathcal{R}^1\left(\overline{V}^1\right) & \approx (0.384, 1.024) \\ \nabla \mathcal{R}^2\left(\underline{V}^2\right) & = (0.5, 0.5) \end{cases} \Rightarrow \mathcal{S}_1^1 > \mathcal{S}_1^2, \quad \text{and} \begin{cases} \nabla \mathcal{R}^2\left(\overline{V}^2\right) & \approx (0.123, 1.294) \\ \nabla \mathcal{R}^1\left(\underline{V}^1\right) & = (0.75, 0.25) \end{cases} \Rightarrow \mathcal{S}_2^1 > \mathcal{S}_2^2,$$

It follows by Theorem 2 (i) that consumer 1 dominates the market almost surely. Note that the effective beliefs of consumer 1 when she dominates the market add up to more than one, which can be interpreted as the consumer becoming endogenously more patient. This property is necessary for the consumer with incorrect beliefs to dominate the market.[19]

Remark: In these examples, consumers have identical discount factors and so local dominance depends only on their effective beliefs. It turns out that for a non-expected utility consumer i, her effective beliefs differ from their actual beliefs in three senses:

– First, the odds of the states are distorted because the effective beliefs over states are obtained by re-weighting the original beliefs. In Example 13, consumers increase the weight on low endowment states when they consume all the endowment, which in some cases make the effective belief of the more optimistic consumer to be closer to the true distribution.

– Second, the total mass of future states is distorted as it can add up to more or less than one, which can be interpreted as the consumer becoming endogenously more patient or more impatient, respectively. In Example 14, the effective beliefs of the consumer with incorrect beliefs, when she consumes all the endowment, put mass sufficiently larger than one, compensating for her incorrect effective odds.

– Third, they are allocation dependent and might cause a reversal of the local dominance relation, allowing the existence of multiple survivors as in Example 13.

6.2 Multiple-consumer case

In this section, we extend the previous section characterization to general cases with an arbitrary number of consumers. We also argue that the survival patterns with multiple consumers cannot

[19] Although the effective beliefs of consumer 2 when she dominates the market also satisfy the necessary condition, they add up to a smaller number and are more skewed leading to a much lower survival index.

be considered as a simple extension of the two-consumer case.

We first provide a condition for a consumer to dominate the market with positive probability.

Proposition 2. *Suppose that Assumptions 1 to 6 hold. Then for every $i \in I$, we have: (i) If there exists some $j \in I$ such that $j \succ_i i$, then consumer i dominates the market with $\mathbb{P}$-zero probability; (ii) if Assumption 7 holds and $i \succ_i j$ for all $j \neq i$, then consumer i dominates the market with a $\mathbb{P}$-strictly positive probability.*

Proposition 2 provides a full characterization of the property that some consumer dominates with a positive probability: A consumer dominates the market with a strictly positive probability if—and only if except for tie cases—she locally dominates all other consumers. It's worth noting that Proposition 2 identifies consumers who dominate the market but does not fully address which consumers will survive or vanish. Next, we demonstrate that in some cases, we can even identify the complete set of survivors.

Definition 6. (Local contour set) For each i, we define *i's* ***local upper contour set*** $U_i = \{j : j \succ_i i\}$ and i's ***local lower contour set*** $D_i = \{j : i \succ_i j\}$.

We impose the following assumption.

Assumption 8. *(No-indifference) For all i and $j \neq i$, either $j \in U_i$ or $j \in D_i$ holds.*

The local upper (lower) contour set of consumer i consists of the consumers who dominate i (are dominated by i) when i consumes the aggregate endowment. Assumption 8 stipulates that every consumer belongs to either the local upper contour set or the lower contour set of any other consumer, thereby ruling out indifference cases. The following theorem uses Assumption 8.

Theorem 3. *Suppose that Assumptions 1 to 8 hold, and that there exists a re-labeling of consumers $\{i_1, i_2, ..., i_n\}$ such that*

$$U_{i_1} \subset U_{i_2} \subset ... \subset U_{i_n}.$$

Let $I^* = \{i : U_i = \emptyset\}$, then (i) all consumers not in I^* will vanish $\mathbb{P}$-almost surely, and (ii) every consumer in I^* will dominate the market with a $\mathbb{P}$-strictly positive probability.

Theorem 3 states that if every consumer's local upper contour set can be ranked completely based on set inclusion, then survivors are the consumers whose local upper contour set is empty, indicating that they are locally undominated. Each of these survivors will dominate the market with a strictly positive probability. Figure 3 provides some illustrative examples. In Figure 3 (a), we have

$$U_1 = \{2, 3\}, U_2 = \{3\}, \text{and } U_3 = \emptyset,$$

resulting in a complete ranking: $U_3 \subset U_2 \subset U_1$. According to Theorem 3, the only survivor is $I^* = \{3\}$, meaning that consumer 3 almost surely dominates the market, while consumers 1 and 2 almost surely vanish. In Figure 3 (b), we have

$$U_1 = \{2, 3\}, U_2 = U_3 = \emptyset,$$

resulting in a set of survivors: $I^* = \{2, 3\}$. In this case, consumer 1 will vanish almost surely, and consumer 2 and 3 will dominate the market, each with a strictly positive probability.

To explain the proof, we will focus on the scenario depicted in Figure 3 (a) and demonstrate that consumer 3 almost surely dominates. First, we note that under the irreducibility assumption, the marginal utility ratio $L_t = \left(L_t^{ij}\right)$ must repeatedly enter at least one of the "one-consumes-all" neighbourhoods infinitely often. Let's suppose that L_t enters the neighborhood of consumer 1. Within consumer 1's neighborhood, we have $U_1 = \{2, 3\}$, and according to Proposition 2, consumer 1 cannot dominate the market. Therefore, L_t will almost surely escape from consumer 1's neighborhood. Once L_t is outside of consumer 1's neighborhood, one of two things must occur: either consumer 2 or consumer 3's consumption level becomes greater than some positive constant. In accordance with Assumption 7, this implies that, after a bounded number of steps, L_t will enter the neighborhood of either consumer 2 or consumer 3. Suppose L_t enters the neighborhood of consumer 2. Again, using the information that $U_2 = \{3\}$, we can conclude that L_t will almost surely escape from consumer 2's neighborhood and enter the neighborhood of consumer 3 with a positive probability. Once L_t enters consumer 3's neighborhood, we have $U_3 = \emptyset$, and Proposition 2 implies that L_t will be trapped in that neighborhood with a positive probability. In summary, L_t will visit consumer 3's neighborhood infinitely often, and each visit will be captured with a positive probability. Consequently, L_t will eventually settle in consumer 3's neighborhood, and as a result, consumer 3 almost surely dominates.

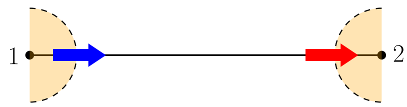
There are situations where Theorem 3 doesn't apply. Figure 3 (c) shows one example, where we can't rank the upper contour sets completely. In this scenario, we can apply Proposition 2 to demonstrate that no consumer will dominate, meaning that at least two consumers will survive. However, we are still unable to determine the identity of the survivors. The following theorem allows us to provide more insight in such cases.

Theorem 4. *Suppose that Assumptions 1 to 8 hold. Let $G \subset I$ be any subset satisfying that $\{U_i \cap G : i \in G\}$ can be completely ranked by set inclusion, and let $G^* = \{i \in G : U_i \cap G = \emptyset\}$. If $U_i \neq \emptyset$ for all $i \in G^*$, then $\mathbb{P}$-almost surely some consumer in $I \backslash G$ survives.*

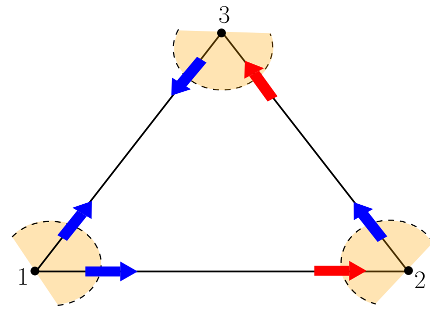
Applying Theorem 4 to Figure 3 (c), we can further conclude that both consumers 2 and 3 will almost surely survive. To see that, we first choose $G = \{1, 3\}$. Note that

$$U_1 \cap G = \{3\}\,, \text{and } U_3 \cap G = \emptyset,$$

so they can be ranked by set inclusion, and we obtain $G^* = \{3\}$. Now, since $U_3 = \{2\} \neq \emptyset$, Theorem 4 implies consumers in $I \backslash G = \{2\}$ survives almost surely. The survival of consumer 3 is symmetric by letting $G = \{1, 2\}$. Here's the intuition: When all consumers in $I \backslash G$ vanish, the economy can be approximated as one with only consumers in G. Theorem 3 implies that in an economy with G, some consumer in G^* must dominate except for null events. However, since $U_i \neq \emptyset$ for every $i \in G^*$, Proposition 2 implies that no one in G^* can dominate. Therefore, the event of everyone in



(a): consumer 1 vanishes a.s.



(b): consumer 1 survives a.s.

Figure 4: Revitalization of an extinct consumer

$I \backslash G$ vanishing must have a probability of 0.

Remark 1. Theorems 3 and 4 nest Theorem 2 as a special case. More specifically, Theorem 3 implies Theorem 2 (i) and (iii), in which the upper contour sets can be ranked; Theorem 4 implies Theorem 2 (ii), in which $U_i = \{j\}$ and $U_j = \{i\}$, so we can choose G to be any singleton to show that $I \backslash G$ almost surely survives.

It is worth noting that the multiple-consumer case can't be considered as a straightforward extension of the two-consumer case. The following example, using Theorem 4, illustrates that it's possible for a consumer who almost surely vanishes in a two-consumer economy to almost surely survive in a larger economy.

Example 15. Consider the same setup as in Example 13. We have the following cases.

- *Consumer 1 vanishes a.s. in two-person economy.* Suppose the economy originally has two consumers $I = \{1, 2\}$ with subjective beliefs $\pi^1 = (1/3, 2/3)$, $\pi^2 = (3/4, 1/4)$, and ambiguity aversion is $\gamma_1 = \gamma_2 = \ln 2$. From the discussion in Example 13, we know $2 \succ_1 1$ and $2 \succ_2 1$ as depicted in Figure 4 (a), so consumer 1 vanishes almost surely.

- *Consumer 1 survives a.s. in three-person economy.* Now consider a new economy $I = \{1, 2, 3\}$ by adding a consumer 3 with $\pi^3 = (1/2, 1/2)$ and $\gamma_3 = \ln(7/3)$. It can be verified that effective beliefs are
$$\nabla \mathcal{R}^i \left(\overline{V}^i \right) = \begin{cases} (0.2, 0.8) & i = 1 \\ (0.6, 0.4) & i = 2 \\ (0.3, 0.7) & i = 3 \end{cases} \quad \text{and} \quad \nabla \mathcal{R}^i \left(\underline{V}^i \right) = \pi^i,$$
Then, it is then easy to verify that local dominance relations are as depicted in Figure 4 (b). Consider a subset $G = \{2, 3\}$. Then, we note that
$$\emptyset = U_3 \cap G \subset U_2 \cap G = \{3\},$$
so $G^* = \{3\}$. Besides, $U_3 \neq \emptyset$, so Theorem 4 implies that consumer 1 survives almost surely.

This example shows that the analysis of survival of a two-consumer economy with general preferences in the literature (e.g. Guerdjikova and Sciubba (2015) and Borovička (2020)) is not sufficient to understand the prospects for survival in general. With general preferences, the local survival indexes can be allocation dependent, leading to a reversal of the indexes' order. In our example above, the local survival index of consumer 2 is larger than that of consumer 1 when either of them dominates the market (so consumer 1 would vanish in a two-consumer economy with $I = \{1, 2\}$) and the local survival index of consumer 3 is larger than that of consumer 2 (so consumer 2 would vanish in a two-consumer economy with $I = \{2, 3\}$). However, consumer 1's local survival index is the largest when consumer 3 dominates the market, which prevents consumer 1 from vanishing in the three-consumer economy. This reversal of local survival indexes cannot occur when all consumers have SEU preferences. Therefore, in the SEU case, a consumer who vanishes in a two-consumer economy must also vanish in a larger economy.

7 Non-robustness of the Market Selection Hypothesis

This section introduces our second main result. It examines the robustness of the market selection hypothesis. In the SEU case, if all consumers have the same discount rate, those with correct beliefs dominate the market. The natural question is whether this result is robust: If we allow consumers to have general preferences, do the consumers with SEU preference with correct beliefs still dominate? If not, can we find a particular preference relation with which a consumer always dominates the market? Before answering this question, we impose the following assumptions.

Assumption 9. *For all $t \in \mathbb{T}$ and $\sigma \in \Sigma$, there exists $s, s' \in S$ such that $e(\sigma_t, s) \neq e(\sigma_t, s')$.*

Assumption 10. *For all $i \in I$, $F^i(c, y) = u(c) + \beta y$ for some $\beta \in (0, 1)$.*

Assumption 9 requires that the aggregate endowment is uncertain. Without uncertainty, consumers will act as if they have SEU preferences locally, and the problem degenerates to the standard case. Assumption 10 requires that all consumers have the same separable time aggregator. Assumption 10 is imposed because: First, to make the discussion meaningful, we must require that all consumers discount the future in the same manner, i.e., we need to require all F^i to be equal, because otherwise we can always make a consumer more likely to survive by making her more patient;[20] second, the separable time aggregator nests the standard discounted expected utility, which enables us to examine the robustness of the standard market selection results. In this section, we allow $\mathcal{R} \in \mathscr{R}$, where $\mathscr{R}$ stands for all aggregators satisfying Assumptions 2, 5 and 6. Note that under Assumption 10, preference differences stem solely from varying attitudes towards uncertainty, described by the certainty equivalent aggregator $\mathcal{R}$. Therefore, we refer to $\mathcal{R}$ as the preferences of the consumers throughout this section for simplicity. In what follows, we are

[20] The key element of Assumption 10 is that the discount rates are all equal to β. Assumption 10 also requires that consumers have a homogeneous utility function u, because it simplifies notation so that we can only use $\mathcal{R}$ to capture preferences differences, but the results won't change if we allow for heterogeneous u.

interested in examining whether there exists some $\mathcal{R} \in \mathscr{R}$ with which a consumer always dominates the market.

7.1 The ε-perfection of the economy

It is worth noting that our characterizations in Theorems 2, 3 and Proposition 2 may not apply to all preferences within $\mathscr{R}$. More specifically, whether Assumption 7 holds depends on various details of the economy, such as the state space, distribution, and endowment. Fortunately, we have found that for every preference $\mathcal{R} \in \mathscr{R}$ and every economy, Assumption 7 can be satisfied by introducing a small ε-perturbation to it. This perturbed economy is referred to as the ε-perfected economy. By focusing on perfected economies, we can discuss all preferences in $\mathscr{R}$ in a unified manner. The following section describes how to construct this perturbation. This construction primarily serves technical purposes, so first-time readers may opt to skip this part.

Construction of the ε-perfection. Recall that Assumption 7 fails when some consumer's consumption decreases in every state, preventing us from reaching the neighborhood where this consumer almost dominates the market. However, if there exists at least one state in which this consumer's consumption increases, even if that state occurs with low probability, it is sufficient to restore Assumption 7. The idea behind the ε-perturbation is to introduce such low-probability states that enable consumption dynamics to enter each "consume-all neighborhood". Below is the formal definition.

Let tuple $\mathcal{E}_S = (e, \pi_0, F, \mathcal{R})$ denote the original economy, where $e = \{e_i\}_{i\in I}$, $F = \left\{F^i\right\}_{i\in I}$, and $\mathcal{R} = \left\{\mathcal{R}^i\right\}_{i\in I}$ denote the profiles of endowment, time aggregators, and certainty equivalent aggregators, and the subscript S emphasizes that the state space is S.[21] Consider another economy with an extended state space $\hat{S} \supset S$ and tuple $\mathcal{E}_{\hat{S}} = \left(\hat{e}, \hat{\pi}_0, \hat{F}, \hat{\mathcal{R}}\right)$. Next we want to talk about the distance between $\mathcal{E}_{\hat{S}}$ and $\mathcal{E}_S$. Note that $\mathcal{E}_S$ is defined on a smaller domain, but we can describe all elements $\mathcal{E}_S$ in the same domain as $\mathcal{E}_{\hat{S}}$ by natural extension, i.e., putting zeros in additional dimensions. The distance $||\mathcal{E}_S - \mathcal{E}_{\hat{S}}||$ is then defined as the maximum distance between the coordinates of $\mathcal{E}_{\hat{S}}$ and $\mathcal{E}_S$ on the extended domain. The formal definition of the distance is stated in Appendix A.11.

Definition 7. We say that $\mathcal{E}_{\hat{S}}$ is an ε***-perfection*** of $\mathcal{E}_S$ if we have: (i) $||\mathcal{E}_S - \mathcal{E}_{\hat{S}}|| < \varepsilon$, and (ii) $\mathcal{E}_{\hat{S}}$ satisfies Assumptions 1 to 7.

An ε-perfected economy is associated with an extended state space that satisfies two key conditions. First, it provides an ε-approximation to the original economy, meaning that (i) the characteristics of the new economy, such as preferences and distributions, are ε-close to the old ones, and (ii) the new states are negligible, occurring with at most ε probability, carrying at most ε endowment, and having negligible effects on preferences. Second, it must also satisfy

[21] Throughout this section, the discount rate for all consumers is equal to β (Assumption 10) and the set of consumers is always I, so we remove them from the definition of an economy for brevity.

the assumptions required for our characterizations. The existence of an ε-perfected economy is guaranteed by the lemma below.

Lemma 2. (Existence) *For all $\mathcal{E}_S$ satisfying Assumptions 1 to 6 and all $\varepsilon > 0$, there exists an ε-perfection of $\mathcal{E}_S$.*

Lemma 2 says that for all the economies of interest, we can always find a perfected economy that is arbitrarily close to the original. This lemma, therefore, enables us to apply this paper's characterizations in a universal manner. It can be applied directly to any given economy or to a small perturbation of it. Moreover, the perturbed economy can approximate the original preferences arbitrarily well, enabling us to explore the robustness of survival for a wide range of preferences examined in this paper.

7.2 Global non-robustness of the MSH

We begin our discussion by examining the robustness of the market selection hypothesis at a global level, encompassing all preferences in $\mathscr{R}$. To facilitate the discussion, we introduce the following key concepts.

Definition 8. (Robust dominator) Preference $\mathcal{R}^i \in \mathscr{R}$ ***robustly dominates*** if for all $\left\{\mathcal{R}^j\right\}_{j\neq i} \in \mathscr{R}^{I-1}$ with $\mathcal{R}^j \neq \mathcal{R}^i$ for all $j \neq i$, consumer i dominates $\mathbb{P}$-almost surely in all ε-perfected economies with sufficiently small ε.[22]

This definition implies that preference $\mathcal{R}^i$ dominates the market in a robust sense. In other words, a consumer with $\mathcal{R}^i$ dominates almost surely, regardless of other consumers' preferences, in all perfected economies that are sufficiently close to the original economy.

Definition 9. (Robust survivor) Preference $\mathcal{R}^i \in \mathscr{R}$ ***robustly survives*** if for all $\left\{\mathcal{R}^j\right\}_{j\neq i} \in \mathscr{R}^{I-1}$ with $\mathcal{R}^j \neq \mathcal{R}^i$ for all $j \neq i$, consumer i survives with a $\mathbb{P}$-strictly positive probability in all ε-perfected economies with sufficiently small ε.

Preference $\mathcal{R}^i$ robustly survives if, regardless of other consumers' preferences, consumer i survives with a strictly positive probability in all perfected economies with small perturbations. This implies that $\mathcal{R}^i$ cannot robustly vanish in the market. It's important to note that $\mathcal{R}^i$ cannot robustly dominate if there is another preference that robustly survives.

Example 16. (Expected-utility case) Suppose that we restrict attention to SEU preferences with i.i.d. beliefs, $\mathscr{R}_{EU}$, where

$$\mathscr{R}_{EU} = \left\{\mathcal{R} : \mathcal{R}\left(V\right) = \mathbb{E}_{\pi}\left(V\right), \text{ where } \pi \in \Delta\left(S\right)\right\},$$

In the space of SEU preferences, the preference with correct belief, $\mathcal{R}_0 = \mathbb{E}_{\pi_0}\left(V\right)$, is the unique robust dominator and survivor. This is because for all profile of beliefs $\pi^{-i} \equiv \left\{\pi^j\right\}_{j\neq i}$ that consists

[22] Formally, there exists some $\delta > 0$ such that consumer i dominates $\mathbb{P}$-a.s. in all ε-perfected economies with $\varepsilon < \delta$.

of incorrect beliefs, the consumer with the correct belief has a strictly higher survival index. Due to the continuity and strict local dominance relation, she also has a strictly higher survival index, and hence will dominate the market, in all ε-perfected economies with sufficiently small ε.

In summary, on $\mathscr{R}_{EU}$, there exists a unique preference that robustly dominates. However, the argument fails when we extend the discussion to the general preference space $\mathscr{R}$.

Theorem 5. *Under Assumptions 1 to 10, (i) there doesn't exist an $\mathcal{R}^i \in \mathscr{R}$ that robustly dominates; (ii) there exist multiple $\mathcal{R}^i \in \mathscr{R}$ that robustly survive; (iii) $\mathcal{R}^i$ robustly survives **if** $i \succ_i r$ and **only if** $i \succeq_i r$, where r denotes the consumer with the correct expected-utility preference.*

Theorem 5 conveys two important messages regarding the validity of the market selection hypothesis. First, there is *no* preference that can robustly dominate in the general preference space $\mathscr{R}$, indicating that MSH is not robust with general preferences. Second, there are *multiple* preferences that can robustly survive within $\mathscr{R}$, suggesting that the survival of heterogeneous preferences is a robust result under our general preference framework. Additionally, Theorem 5 also provides a simple characterization of robust survivors: a preference relation robustly survives if and almost only if any consumer with those preferences locally dominates the rational consumer (i.e., the correct SEU consumer)—so the set of robust survivors are those who include the correct SEU preferences in the local lower contour set.

Proof sketch of Theorem 5

Theorems 5 (i) and (ii) are proved by construction, and we will provide an example to illustrate this later (see Example 17). The main focus is to prove Theorem 5 (iii). To prove it, we first show that under our assumptions, preferences must exhibit the ***local-expected-utility property*** at certainty, which means that effective beliefs must be a probability distribution if the continuation utility is constant across all states.

Lemma 3. *(Local-EU property) For all $\mathcal{R} \in \mathscr{R}$ and $k \in \mathbb{R}_{++}$, we have $\nabla \mathcal{R}(k \cdot \mathbf{1}) \in \Delta(S)$.*

Proof. Euler's homogeneous function theorem implies that for all $k \in \mathbb{R}_{++}$, we have

$$k = \mathcal{R}(k \cdot \mathbf{1}) = k \times \frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial k} = k \times \sum_{s \in S} \frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial V_s}$$
$$\Rightarrow \sum_{s \in S} \frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial V_s} = 1.$$

Assumption 1 says that $\frac{\partial \mathcal{R}(k \cdot \mathbf{1})}{\partial V_s} > 0$, so $\nabla \mathcal{R}(k \cdot \mathbf{1})$ is a probability distribution. $\square$

Next we sketch the proof of Theorem 5 (iii). For simplicity, we focus on the two-consumer case, $I = \{i, j\}$. To prove the **"if"** direction, suppose that $i \succ_i r$. In cases where consumer i consumes all the endowment, consumer j bears no uncertainty and behaves like an expected-utility consumer due to Lemma 3. Therefore, we must have $r \succeq_i j$ because any SEU consumer is locally dominated by the correct SEU consumer. Transitivity then implies $i \succ_i j$, which further leads to the conclusion that consumer i dominates the market with a strictly positive probability by Theorem 2. Importantly, this argument holds irrespective of consumer j's preference, making consumer i a robust survivor. To prove the **"only if"** direction, suppose that $r \succ_i i$. In this scenario, we can construct a preference $\mathcal{R}^j \in \mathscr{R}$ such that: (i) Consumer j locally dominates the rational consumer r, and (ii) Consumer j's effective belief becomes correct when she vanishes, e.g., the HAAA preference in Example 17 with the correct belief. Lemma 3 implies that when consumer j dominates, consumer i will act like an SEU consumer. Therefore, (i) implies that $j \succ_j i$. Similarly, when consumer i dominates, (ii) implies that consumer j acts like an SEU consumer with the correct belief. Consequently, if $r \succ_i i$, it implies $j \succ_i i$. Since consumer j dominates consumer i in both neighborhoods, consumer i vanishes almost surely as per Theorem 2. Consequently, consumer i cannot be a robust survivor. The following example uses Theorem 5 to demonstrate how to identify robust survivors.

Example 17. (Robust survivors with HAAA) Consider the case where $S = \{s_1, s_2\}$, $\pi_0 = (1/2, 1/2)$, and a consumer with HAAA preferences where

$$V_{\sigma_t} = u(c_t) + \beta \times \phi^{-1}\left(\mathbb{E}_\pi \phi\left(V_{\sigma_t,s}\right)\right), \quad \text{where } \phi(x) = \frac{1-\gamma}{\gamma}\left(\frac{x}{1-\gamma} + m\right)^\gamma.$$

For simplicity, let's suppose $m = 0$ and $\beta \approx 0$. Let $u_s \equiv u(e_s)$ denote the utility value of the aggregate endowment in state s, then effective beliefs are

$$\nabla_s \mathcal{R}\left(\overline{V}\right) \approx \frac{u_s^{\gamma-1} \times \pi(s)}{\left(\sum_{s\in S} u_s^\gamma \times \pi(s)\right)^{1-1/\gamma}}, \text{ and } \nabla_s \mathcal{R}\left(\underline{V}\right) = \pi(s).$$

From Theorem 5, the preference robustly survives if

$$\mathbb{E}\log \frac{u_s^{\gamma-1} \times \pi(s)}{\left(\sum_{s\in S} u_s^\gamma \times \pi(s)\right)^{1-1/\gamma}} > \mathbb{E}\log \pi_0(s). \tag{17}$$

and only if (17) holds with weak inequality. Figure 5 visually represents the set of robust survivors, with $u_1 = 1$, $u_2 = \exp(k) > 1$, and the vertical axis denoting $\pi(s_1)$. We see that: (i) robust surviving preferences exist when $\gamma \in (0,1)$, and (ii) for any $\gamma \in (0,1)$, as the aggregate uncertainty increases $k \to +\infty$, the set of robustly-surviving beliefs π expands to the whole interval $[0,1]$. That is, *every* full-support belief can robustly survive under some HAAA preference if there is sufficiently high uncertainty.

7.3 Local non-robustness of the MSH

The market selection hypothesis is not only globally but also locally robust within the class of SEU preferences in the sense that incorrect beliefs can't survive in the presence of some arbitrarily close

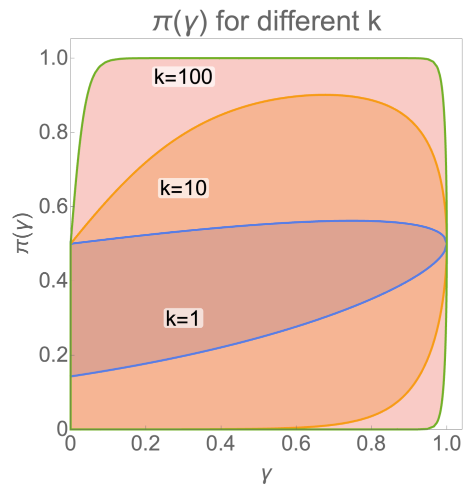


Figure 5: Robust survivors with HAAA preferences

beliefs. In this section, we further show the local robustness property does not extend to the case of general preferences. To discuss the local robustness, we need to define some distance notion. Let $\mathcal{V}$ denote a compact set that contains all possible utility values.[23] We define

$$D_0\left(\mathcal{R},\mathcal{R}'\right)=\max_{V\in\mathcal{V}}\left|\mathcal{R}\left(V\right)-\mathcal{R}'\left(V\right)\right| \text{ and } D_1\left(\mathcal{R},\mathcal{R}'\right)=\max_{V\in\mathcal{V},s\in S}\left|\nabla_s\mathcal{R}\left(V\right)-\nabla_s\mathcal{R}'\left(V\right)\right|,$$

and define $D=\max\left\{D_0,D_1\right\}$. We adopt metric D throughout this subsection, but all metrics equivalent to (or finer than) D also work. We believe that D serves as a natural distance notion as it captures the idea that if two preferences are similar, their utility and marginal utility levels should also be similar. We introduce the following concept.

Definition 10. (Locally robust dominator) Preference $\mathcal{R}^i\in\mathscr{R}$ ***locally robustly dominates*** if there exists some $\delta>0$ such that for all $\left\{\mathcal{R}^j\right\}_{j\neq i}\in\mathscr{R}^{I-1}$ with

$$0<D\left(\mathcal{R}^j,\mathcal{R}^i\right)<\delta,\quad\text{for all } j\neq i,$$

consumer i dominates $\mathbb{P}$-almost surely in all ε-perfected economies with sufficiently small ε.

This definition differs from the previous definition of a robust dominator by restricting all other preferences to be within distance δ of $\mathcal{R}^i$. In simple terms, preference $\mathcal{R}^i$ locally robustly dominates if consumer i always dominates the market in the presence of sufficiently similar preferences. Recall that Theorem 5 states that no preference can robustly dominate. A natural conjecture is that some preferences might robustly dominate in a local sense. The following theorem refutes this conjecture.

Theorem 6. *Under Assumptions 1 to 10, there is no preference $\mathcal{R}^i\in\mathscr{R}$ that locally robustly dominates.*

[23]It is well-defined because the endowment is bounded, e.g., we can let $\mathcal{V}=[0,\overline{v}]$ with $\overline{v}=\max_i\frac{u_i(\overline{e})}{1-\beta}$ where $\overline{e}$ is maximum level of total endowment.

Theorem 6 asserts that no preference can robustly dominate in a local sense. Specifically, if $\mathcal{R}^i$ is the expected-utility preference with correct belief, Theorem 6 implies that there exists another preference arbitrarily close to it that prevents it from dominating the market almost surely, which shows that the standard result that a correct SEU consumer dominates the market is not robust. Before explaining the proof, let's first introduce the concept of locally robust survivor.

Definition 11. (Locally robust survivor) Preference $\mathcal{R}^i \in \mathscr{R}$ ***locally robustly survives*** if there exists some $\delta > 0$ such that for all $\left\{\mathcal{R}^j\right\}_{j \neq i} \in \mathscr{R}^{I-1}$ with

$$0 < D\left(\mathcal{R}^j, \mathcal{R}^i\right) < \delta, \quad \text{for all } j \neq i,$$

consumer i survives with a $\mathbb{P}$-strictly positive probability in all ε-perfected economies with sufficiently small ε.

Similarly, a preference relation locally robustly survives if a consumer with those preferences survives almost surely in the presence of consumers with preferences within some small neighborhood. Let's look at the SEU case for a better understanding.

Example 18. (Local robustness with SEU) As in Example 16, we study the preferences that locally robustly dominates/survives on $\mathscr{R}_{EU}$. It can be verified that $\mathcal{R}_0 = \mathbb{E}_{\pi_0}(V)$ is also the unique preference that locally robustly dominates or survives. This is because for any other belief $\pi \neq \pi_0$, there exists another belief $\hat{\pi}$ that has a lower relative entropy and can be arbitrarily close to π.[24] This implies that a consumer with any incorrect beliefs will be driven out of the market by another consumer with an arbitrarily close belief, so the consumer can't be a locally robust dominator or survivor, which shows the local robustness of MSH within SEU.

Next, we discuss how to find preferences that locally robustly survive in a general class of preferences. A natural conjecture is that we should have a characterization parallel to Theorem 5, where a preference relation locally robustly survives if and (almost) only if a consumer with those preferences locally dominates an expected-utility consumer with some local belief. However, the characterization turns out to be very different from Theorem 5. To present the characterization, we introduce the following concepts.

Definition 12. Preference $\mathcal{R}^i \in \mathscr{R}$ is ***effectively uncertainty neutral*** if

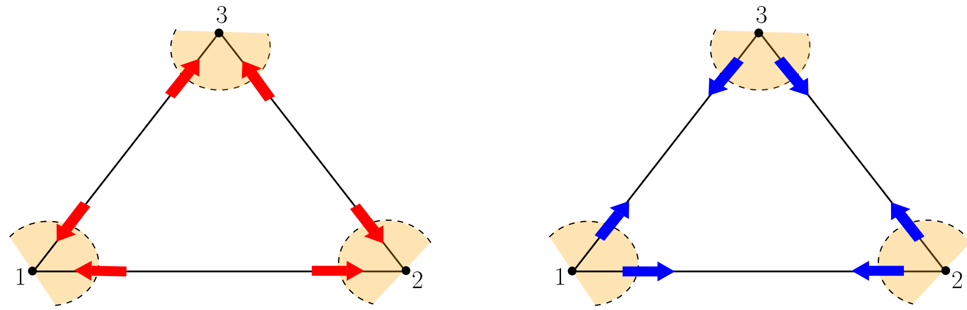
$$\mathbb{E} \log \left[\nabla_s \mathcal{R}^i\left(\underline{V}^i\right)\right] = \mathbb{E} \log \left[\nabla_s \mathcal{R}^i\left(\overline{V}^i\right)\right], \tag{18}$$

and is ***effectively uncertainty favored*** (or ***unfavored***) if the L.H.S. of (18) is less (or greater) than the R.H.S.

[24] For example, we define $\hat{\pi} = (1-\varepsilon)\pi + \varepsilon\pi_0$ where $\varepsilon \in (0,1)$. The strict convexity of the relative entropy implies that

$$I(\hat{\pi}) < (1-\varepsilon) I(\pi) + \varepsilon I(\pi_0) = (1-\varepsilon) I(\pi) < I(\pi),$$

so $\hat{\pi}$ has a smaller entropy and can be arbitrarily close to π if we take ε sufficiently small.



(a): uncertainty favored (b): uncertainty unfavored

Figure 6: Locally robust survivor

In simple terms, $\mathcal{R}^i$ is effectively uncertainty neutral if the effective beliefs when i dominates and when i vanishes have the same survival index. Effectively uncertainty-neutral consumers can be loosely thought of as SEU consumers. This is because with discounted SEU preferences, effective beliefs are allocation-independent, so (18) always holds; in contrast, in most cases with non-expected utility, effective beliefs are allocation-dependent, and are often different when a consumer bears all uncertainty compared to when she bears no uncertainty. In cases where the effectively uncertainty neutrality is absent, we refer to a consumer as effectively uncertainty favored (or unfavored) if their local survival index is greater when they consume all the endowment compared to when they consume nothing.

Theorem 7. *Suppose that $|I| = 2$. Under Assumptions 1 to 10, $\mathcal{R}^i \in \mathscr{R}$ locally robustly survives if it is not effectively uncertainty neutral.*

By Example 18, we know that an expected-utility consumer with any incorrect belief can't be a locally robust survivor. However, Theorem 7 presents a sharp contrast by showing that a consumer with any preference that is not effectively SEU (i.e., effectively uncertainty neutral) is a locally robust survivor. Theorem 7 establishes that for a class of general preferences, the local selection result with SEU preferences represents a knife-edge case. Furthermore, it suggests that the co-existence of consumers seems more robust—*every* consumer who is not effectively neutral towards uncertainty can robustly survive at least in a local sense.

Sketch of the proof of Theorems 6 and 7.

We'll begin by discussing how to prove Theorem 6. For each $\mathcal{R}^i \in \mathscr{R}$, there are three possibilities: $\mathcal{R}^i$ can be effectively uncertainty neutral, favored, or unfavored. We will demonstrate that $\mathcal{R}^i$ does not locally robustly dominate in any of these cases.

(i) $\mathcal{R}^i$ is effectively uncertainty favored. In this case, for all preferences sufficiently close to $\mathcal{R}^i$, the local dominance pattern is depicted as Figure 6 (a), where every consumer locally dominates all other consumers. According to Proposition 2, every consumer dominates the market with a strictly positive probability. Therefore, $\mathcal{R}^i$ can't locally robustly dominate.

(ii) $\mathcal{R}^i$ is effectively uncertainty unfavored. In this case, for all preferences sufficiently close to $\mathcal{R}^i$, the local dominance pattern is depicted as Figure 6 (b), where every consumer is locally dominated by all other consumers. Proposition 2 implies that no consumer can dominate the market. This, in turn, implies that $\mathcal{R}^i$ cannot locally robustly dominate.

(iii) $\mathcal{R}^i$ is effectively uncertainty neutral. In this case, we can construct a preference $\mathcal{R}^j$ arbitrarily close to $\mathcal{R}^i$ that prevents i from dominating the market. One example is given by:

$$\mathcal{R}^j \equiv (1-\delta)\,\mathcal{R}^i + \delta\mathcal{R}^k,$$

where $\nabla\mathcal{R}^k$ satisfies has a lower entropy than $\nabla\mathcal{R}^i$ at $\overline{V}^i$ (roughly, we can think of it as $k \succ_k i$).[25] Under this construction, for all $\delta > 0$, we can show that $j \succ_j i$ in some ε-perfected economy for any $\varepsilon > 0$, so $\mathcal{R}^i$ can't locally robustly dominate the market.

The proof of Theorem 7 follows directly. As explained above, we have two cases: (i) When $\mathcal{R}^i$ is effectively uncertainty favored, every consumer dominates the market with a positive probability, so $\mathcal{R}^i$ locally robustly survives; (ii) When $\mathcal{R}^i$ is effectively uncertainty unfavored, every consumer is locally dominated by all other consumers. In the case of $|I| = 2$, each consumer survives almost surely as per Theorem 2. Therefore, $\mathcal{R}^i$ locally robustly survives as well.

Note that Theorem 7 assumes $|I| = 2$. This limitation arises because our characterizations are less comprehensive for multi-consumer cases. Specifically, when $\mathcal{R}^i$ is effectively uncertainty-unfavored as shown in Figure 6 (b), our characterizations don't provide information on which consumer survives. Therefore, we remain agnostic about whether i is a locally robust survivor. However, we do know that at least two consumers coexist on every path, allowing us to establish a weaker but qualitatively similar property.

Definition 13. (Locally robust co-survivor) Preference $\mathcal{R}^i \in \mathscr{R}$ ***locally robustly co-survives*** if for all $\delta > 0$ and $\left\{\mathcal{R}^j\right\}_{j\neq i} \in \mathscr{R}^{I-1}$ with

$$0 < D\left(\mathcal{R}^j, \mathcal{R}^i\right) < \delta, \quad \text{for all } j \neq i,$$

there are *at least two* consumers surviving with a $\mathbb{P}$-strictly positive probability in all ε-perfected economies with sufficiently small ε.

In simple terms, if some preference locally robustly co-survives, then for consumers with preferences in any small neighborhood around $\mathcal{R}^i$, there are at least two consumers who can survive with a strictly positive probability. In other words, no preference can always dominate the market in any neighborhood around $\mathcal{R}^i$. We have the following corollary.

Corollary 1. *Under Assumptions 1 to 10, $\mathcal{R}^i \in \mathscr{R}$ locally robustly co-survives if it is not effectively uncertainty neutral.*

[25] For example, the HAAA preferences with $\gamma \in (0, 1)$ considered in Guerdjikova and Sciubba (2015).

Corollary 1 indicates that multiple consumers can survive in *any* neighborhood around *any* preference that is not effectively uncertainty neutral. In other words, local co-existence is densely spread throughout the general preference space. From previous discussion, the local co-existence can take two forms: When i is effectively uncertainty favored, multiple consumers dominate with positive probability; when i is uncertainty unfavored, multiple consumers co-exist on every path.

8 Conclusion

We provide a unified framework to study the dynamics of competitive equilibrium consumption when markets are complete and consumers' preferences belong to a general class of recursive preferences, nesting both expected-utility and many common non-expected utility preferences as special cases.

Using this framework, we study the robustness of the market selection hypothesis. Sandroni (2000) and Blume and Easley (2006) show that among consumers with SEU, those with correct beliefs dominate the market. Their analysis, however, left open the question of whether this is an intrinsic property of optimal plans made by consumers with correct beliefs or it is due to the ancillary restriction to SEU preferences. Thus, to study whether that property is robust to the introduction of more general preferences, we considered economies with a rich class of recursive preferences. We first characterize when a consumer can robustly survive globally, that is against any combination of consumers with arbitrary preferences in that class. We show that a consumer robustly survives if and only if, except for tie cases, the consumer locally dominates an SEU consumer with correct beliefs. We use our characterization to show there is no preference relation that robustly dominate the market. Later we consider whether a consumer can robustly survive locally, that is against some other consumers with arbitrarily close preferences. We show that any consumer who is not effectively uncertainty neutral is a locally robust survivor. Thus, the local selection result that holds for SEU preferences represents a knife-edge case. Our local and global robustness analysis lets us conclude that, in sharp contrast to the case of SEU, the existence of multiple survivors is the robust long-run outcome when the class of preferences is rich enough.

Our results underscore that there is an inherent tension between the efficiency of competitive equilibrium allocations and the concept of rational expectations equilibrium. Therefore, we give a novel foundation for the idea put-forward by the behavioral finance literature that some irrational behaviors, i.e. incorrect beliefs, can have long-run effects on asset prices. Contrary to the conventional wisdom in economics, we show there is no need to assume the presence of market frictions for such irrationality to have a persistent effect on asset prices in a world with sufficiently rich preferences.

Although we consider general preferences, we have limited ourselves to deviations from expected utility that keep the time consistency property. The relevant case of time-inconsistent preferences is left for future work. Similarly, this paper considers the case of complete markets but one could also use the local approach developed in this paper to deepen our understanding of the dynamics of consumption in incomplete markets economies.

Bibliography

Alchian, A. (1950) "Uncertainty, Evolution and Economic Theory," *Journal of Political Economy*, 58 (1), 211–221. 2

Aliprantis, C.D. and K.C. Border (1999) *Infinite Dimensional Analysis: A Hitchhiker's Guide*: Springer. 42, 43

Beker, P. and S. K. Chattopadhyay (2010) "Consumption Dynamics in General Equilibrium: A Characterization When Markets Are Incomplete," *Journal of Economic Theory*, 145, 2133–2185. 2

Bewley, T. (1969) "A Theorem on the Existence of Competitive Equilibria in a Market With a Finite Number of Agents and Whose Commodity Space is L_∞," in Khan, M. A. and N.C.Yannelis eds. *Equilibrium Theory in Infinite Dimensional Spaces*: Springer. 3, 9, 38

——— (1972) "Existence of equilibria in economies with infinitely many commodities," *Journal of Economic Theory*, 4 (3), 514–540. 38

Blume, L. and D. Easley (1992) "Evolution and Market Behavior," *Journal of Economic Theory*, 74, 9–40. 2

——— (2006) "If You're So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets," *Econometrica*, 74 (4), 926–966. 1, 2, 3, 4, 5, 10, 11, 15, 19, 34

——— (2009) "The Market Organism: Long Run Survival in Markets with Heterogeneous Traders," *Journal of Economic Dynamics and Control*, 33 (5), 1023–1035. 11

Borovička, J. (2020) "Survival and Long-run Dynamics With Heterogeneous Beliefs Under Recursive Preferences," *Journal of Political Economy*, 128 (1), 206–251. 2, 5, 25

Cao, D. (2017) "Speculation and Financial Wealth Distribution Under Belief Heterogeneity," *Economic Journal*, 128, 2258–81. 2

Cerreia-Vioglio, S., D. Dillenberger, and P. Ortoleva (2015) "Cautious Expected Utility and the Certainty Effect," *Econometrica*, 83, 693–728. 7

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and L. Montrucchio and (2011) "Uncertainty Averse Preferences," *Journal of Economic Theory*, 146, 1275â1330. 7

Chew, S. H. (1983) "A Generalization of the Quasilinear Mean With Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox," *Econometrica*, 51, 1065–1092. 6

——— (1989) "Axiomatic Utility Theories With the Betweenness Property," *Annals of Operations Research*, 19, 273–298. 6

Cogley, T. and T. J. Sargent (2009) "Diverse Beliefs, Survival and the Market Price of Risk," *The Economic Journal*, 119 (March), 354–376. 2

Cogley, T., T. J. Sargent, and V. Tsyrennikov (2014) "Wealth Dynamics in a Bond Economy With Heterogeneous Beliefs," *The Economic Journal*, 124 (575), 1–30. 2

Condie, S. (2008) "Living with Ambiguity: Prices and Survival When Investors Have Heterogeneous Preferences for Ambiguity," *Economic Theory*, 36, 81–108. 2, 6

Cootner, P. (1964) *The Random Character of Stock Market Prices*: MIT Press. 2

Coury, T. and E. Sciubba (2012) "Belief Heterogeneity and Survival in Incomplete Markets," *Economic Theory*, 49 (1), 37–58. 2

Cvitanić, J. and S. Malamud (2011) "Price Impact and Portfolio Impact," *Journal of Financial Economics*, 100 (1), 201–25. 2

Da Silva, C. (2011) "On Asymptotic Behavior of Economies With Complete Markets: The Role of Ambiguity Aversion," *Instituto Nacional de Matematica Pura e Aplicada, Rio de Janeiro, Brasil, mimeo*. 2

Dekel, E. (1986) "An Axiomatic Characterization of Preferences Under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory*, 40, 304–318. 6

Dindo, P. (2019) "Survival in Speculative Markets," *Journal of Economic Theory*, 181, 1–43. 2, 5

Easley, D. and L. Yang (2015) "Loss Aversion, Survival and Asset Prices," *Journal of Economic Theory*, 160, 494–516. 2

Epstein, L. G. and S. E. Zin (1989) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57 (4), 937–969. 5

Friedman, M. (1953) *Essays in Positive Economics*: University of Chicago Press. 2

Guerdjikova, A. and E. Sciubba (2015) "Survival With Ambiguity," *Journal of Economic Theory*, 155, 50–94. 2, 6, 12, 25, 33, 56

Gul, F. (1991) "A Theory of Disappointment Aversion," *Econometrica*, 155, 667–686. 6

Hardy, G.H., J.E. Littlewood, and G. Pólya (1952) *Inequalities*, Cambridge Mathematical Library: Cambridge University Press. 9

Klibanoff, P., M. Marinacci, and S. Mukerji (2009) "Recursive Smooth Ambiguity Preferences," *Journal of Economic Theory*, 144 (3), 930–976. 6

Kogan, L., S. A. Ross, J. Wang, and M. M. Westerfield (2006) "The Price Impact and Survival of Irrational Traders," *The Journal of Finance*, 61 (1), 195–229. 2

——— (2017) "Market Selection," *Journal of Economic Theory*, 168, 209–236. 2

Kreps, D. M. and E. L. Porteus (1978) "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200. 6

Krishna, V. and P. Sadowski (2014) "Dynamic Preferences for Flexibility," *Econometrica*, 82 (2), 655–703. 7

Mailath, G. and A. Sandroni (2003) "Market Selection and Asymmetric Information," *Review Economic Studies*, 168, 343–68. 2

Marinacci, M. and L. Montrucchio (2010) "Unique Solutions for Stochastic Recursive Utilities," *Journal of Economic Theory*, 145, 1776–1804. 3, 9, 39, 40, 57

Massari, F. (2019) "Market Selection in Large Economies: A Matter of Luck," *Theoretical Economics*, 14 (2), 437–73. 2

Quiggin, J. (1982) "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization*, 3 (4), 323–343. 7

Routledge, B. R. and S. E. Zin (2010) "Generalized Disappointment Aversion and Asset Prices," *The Journal of Finance*, 65 (4), 1303–1332. 6, 13

Sandroni, A. (2000) "Do Markets Favour Agents Able to Make Accurate Predictions?" *Econometrica*, 68 (6), 1303–1341. 1, 2, 3, 4, 5, 10, 11, 15, 19, 34

Sarver, T. (2018) "Dynamic Mixture-Averse Preferences," *Econometrica*, 86 (4), 1347â1382. 7

Strzalecki, T. (2013) "Temporal Resolution of Uncertainty and Recursive Models of Ambiguity Aversion," *Econometrica*, 81 (3), 1039–1074. 7

Williams, D. (1991) *Probability With Martingales*: Cambridge University Press. 52

Yan, H. (2008) "Natural Selection in Financial Markets: Does It Work?" *Management Science*, 54, 1935–1950. 2

A Proofs

A.1 Proof of Proposition 1

Proof. Since (c, p) is a competitive equilibrium, then $c^i \in \mathbb{C}$ must solve consumer $i's$ maximization problem and, therefore, it must satisfy the one-deviation property. That is, for all $\sigma_t \in \Sigma_t$ and all t, $c(\sigma_t)$ and $\{c(\sigma_t, s) : s \in S\}$ must solve the following problem

$$\begin{aligned} \max \quad & F^i\left(x(\sigma_t), \mathcal{R}_t\left(F^i\left(x(\sigma_t, s), \mathcal{R}^i_{t+1}\left(V^i_{t+2}\right)\right)\right)\right) \\ \text{s.t.} \quad & p(\sigma_t)\, x(\sigma_t) + \sum_{s\in S} p(\sigma_t, s) \times x(\sigma_t, s) \le p(\sigma_t)\, c(\sigma_t) + \sum_{s\in S} p(\sigma_t, s) \times c(\sigma_t, s), \\ & x(\sigma_t), x(\sigma_t, s) \ge 0. \end{aligned}$$

Since Assumptions 1 and 2 hold, then $p(\sigma_t) > 0$ and $c^i(\sigma_t) > 0$ for all σ_t and all i. So the constraint qualification holds and there exists a Lagrange multiplier $\lambda^i(\sigma_t)$ such that $\left(c(\sigma_t), \lambda^i(\sigma_t)\right)$ must satisfy the following FOC:

$$\begin{aligned} \lambda^i(\sigma_t)\, p(\sigma_t) =& F^i_1\left(c^i(\sigma_t), \mathcal{R}^i_t\left(V^i_{t+1}\right)\right) \\ \lambda^i(\sigma_t)\, p(\sigma_t, s) =& F^i_2\left(c^i(\sigma_t), \mathcal{R}^i_t\left(V^i_{t+1}\right)\right) \times \frac{\partial \mathcal{R}^i_t}{\partial V^i_{\sigma_t, s}} \times F^i_1\left(c^i(\sigma_t, s), \mathcal{R}^i_{t+1}\left(V^i_{t+2}\right)\right). \end{aligned}$$

Therefore, we must have

$$\frac{p(\sigma_t, s)}{p(\sigma_t)} = \frac{F^i_1\left(c^i(\sigma_t, s), \mathcal{R}^i_{t+1}\left(V^i_{t+2}\right)\right)}{F^i_1\left(c^i(\sigma_t), \mathcal{R}^i_t\left(V^i_{t+1}\right)\right)} \times \frac{\partial \mathcal{R}^i_t}{\partial V^i_{\sigma_t, s}} \times F^i_2\left(c^i(\sigma_t), \mathcal{R}^i_t\left(V^i_{t+1}\right)\right),$$

□

A.2 Some Useful Results for the Proof of Theorem 1

The economy has countably many commodities, so we can use the following theorem due to Bewley (1969) to establish the existence of a competitive equilibrium in our economy.[26]

Theorem 8. *Let* $E = \left(\mathcal{C}, (C_i, \succeq_i, e_i)_{i\in I}\right)$ *be an economy which satisfies the following:*

1) $\mathcal{C} = L_\infty(B, \mathcal{B}, \upsilon)$ *where* $(B, \mathcal{B}, \upsilon)$ *is a positive measure space.*

2) $\forall i, C_i = x_i + K$, *where* $K = \{c \in \mathbb{C} : c \ge 0\}$ *is the positive cone of* $\mathcal{C}$.

3) $\forall i,\ e_i \gg x_i$,

4) $\forall i$, *the pre-order (i.e. a binary relation that is reflexive and transitive)* $\succeq_i$ *is*

(a) Mackey continuous: for each $\underline{c} \in C_i$, *the sets* $\{c \in C_i : c \succeq_i \underline{c}\}$ *and* $\{c \in C_i : c \precsim_i \underline{c}\}$ *are closed in* C_i *with respect to the Mackey topology for the dual pair* (L_∞, L_1),

[26] The existence theorem in Bewley (1969) focuses on an exchange economy, which can be directly applied here. See Bewley (1972) for an existence theorem that includes production as well.

(b) weakly convex: for each $\underline{c} \in C_i$, the set $\{c \in C_i : c \succeq_i \underline{c}\}$ is convex.

(c) strongly monotone: $c \in C_i$ and $k > 0$ implies that $c + k \succ_i c$,

(d) complete: if $c_1, c_2 \in C_i$, then either $c_1 \succeq_i c_2$ or $c_2 \succeq c_1$.

Then E has a competitive equilibrium.

Next, we introduce the following definitions used by Marinacci and Montrucchio (2010).

Definition 14. F is *γ-subhomogeneous* if there exists some $\gamma > 0$ such that

$$F\left(\alpha^{\gamma} x, \alpha y\right) \geq \alpha F\left(x, y\right)$$

for all $\alpha \in (0, 1]$ and all $x, y \in \mathbb{R}_+$.

Definition 15. F is a *Thompson aggregator* if $F\left(x, 0\right) > 0$ for all $x > 0$, and

$$F\left(x, \alpha y\right) \geq \alpha F\left(x, y\right) + \left(1 - \alpha\right) F\left(x, 0\right)$$

for all $\alpha \in [0, 1]$ and all $x, y \in \mathbb{R}_+$.

Definition 16. $\mathcal{R}$ is *subhomogeneous* if $\mathcal{R}_t\left(\alpha V_{t+1}\right) \geq \alpha \mathcal{R}_t\left(V_{t+1}\right)$ for all $\alpha \in [0, 1]$, $t \geq 0$ and all adapted process $V = \{V_t : t \in \mathbb{T}\}$.

An example of a certainty equivalent function that is subhomogeneous can be found in Example 1. The following fact is easy to verify.

Lemma 4. *Under Assumptions 1, 4 and 5, F is a Thompson aggregator and γ-subhomogeneous, and $\mathcal{R}$ is subhomogeneous.*

Proof. (i) Thompson: From Assumption 1, $F \geq 0$ and $F_1, F_2 > 0$, we must have $F\left(x, 0\right) > 0$ for all $x > 0$; the second condition comes from concavity of F, i.e., Assumption 4. (ii) γ-subhomogeneous: The concavity and non-negativity of F implies that

$$F\left(\alpha x, \alpha y\right) \geq \alpha F\left(x, y\right) + \left(1 - \alpha\right) F\left(0, 0\right) \geq \alpha F\left(x, y\right),$$

then we simply let $\gamma = 1$. (iii) Similarly, $\mathcal{R}$ is subhomogeneous because

$$\mathcal{R}_t\left(\alpha V_{t+1}\right) \geq \alpha \mathcal{R}_t\left(V_{t+1}\right) + \left(1 - \alpha\right) \mathcal{R}_t\left(0\right) \geq \alpha \mathcal{R}_t\left(V_{t+1}\right),$$

where the first equality comes from concavity, and the second comes from non-negativity. $\square$

A.3 Proof of Theorem 1

Let $c \in L_+$ be a consumption plan and $T : L_+ \to L_+$ be the operator given by

$$T\left(V\right) = F\left(c, \mathcal{R}(V)\right) \tag{19}$$

that is, for all $t \geqslant 0$, $T_t(V) = F(c_t, \mathcal{R}_t(V_{t+1}))$. For a fixed consumption sequence $c \in L_\infty$, let T^c be the operator defined in (19). The next lemma shows the existence of a unique $V^c \in L_+$ that solves $V = T^c(V)$ and that the mapping $c \mapsto V^c$ is continuous in the product topology.

Lemma 5. *Under Assumptions 1, 2, 4 and 5, there exists a unique $V^c \in L_+$ that solves equation $V = T^c(V)$ and the mapping $c \mapsto V_t^c$ is pointwise continuous in the product topology for all $t \geq 0$.*

Proof. The existence and uniquenss follows from Theorem 1 in Marinacci and Montrucchio (2010). We only need to show that hypothesis (ii) in their Theorem 10 holds as the proof that hypothesis (i) and (iii) hold is not affected by our assumption that $F(0,0) > 0$. By Assumption 4 there exists y^* such that $F(\|c\|_\infty, y^*) = y^*$. Let $X \in [0, Y^*]$ where $Y_t^* = y^*$ for all $t \geq 0$. Let $\alpha = \frac{F(0,0)}{y^*}$ and $\beta = 1$. Note that $\alpha > 0$ by Assumption 4 and $\beta > 0$. Note also that:

$$\alpha Y^* = F(0,0)1 \leq F(c,0)1 = T(0) \leq Y^* = \beta Y^*$$

where $1 \in L_+$ is the sequence of ones, the first inequality follows because F increases in its first argument by Assumption 1. It follows that condition (ii) holds.

The continuity of the mapping $c \mapsto V_t^c$ in the product topology follows from Theorem 6 in Marinacci and Montrucchio (2010). To conclude the proof we verify that the assumptions of their Theorem 6 hold. Note that F is continuous by assumption 1; besides, it is Thompson and γ-subhomogeneous by Lemma 4. For every consumption plan $c \in L_\infty$, their condition (17) becomes $\lim_{t\to+\infty} F(1,t)/t < 1$. This is satisfied because: (i) by Assumption 4, there is some y such that $F(1,y) = y$, and (ii) by their Lemma 1, $F(1,t)/t$ is strictly decreasing. Again, we know that $\mathcal{R}^i$ is subhomogeneous by Lemma 4. The continuity and monotonicity $\mathcal{R}_t^i$ implies that: (i) $V_{t+1}^n \uparrow V_{t+1}$ implies $\mathcal{R}_t^i\left(V_{t+1}^n\right) \uparrow \mathcal{R}_t^i(V_{t+1})$ for every t, and (ii) $V_{t+1}^n \to V_{t+1}$ implies $\mathcal{R}_t^i\left(V_{t+1}^n\right) \to \mathcal{R}_t^i(V_{t+1})$. So the hypothesis for their Theorem 6 are satisfied. □

Lemma 6. *Under Assumptions 1 and 2, $\succeq_i$ satisfies strong monotonicity.*

Proof. Suppose that $c \geq c'$, that is, $c(\sigma_t) \geq c'(\sigma_t)$ for all $\sigma_t \in \Sigma^t$ and $t \geq 1$. For all $T \geq 1$, we define

$$c^T(\sigma_t) = \begin{cases} c(\sigma_t) & t \leq T \\ c'(\sigma_t) & t > T \end{cases} \quad \text{for all } \sigma_t \in \Sigma^t. \tag{20}$$

Since F is increasing in its first argument, then we have that for every $\sigma_T \in \Sigma^T$:

$$\begin{aligned} V_{\sigma_T}^i\left(c^T\right) &= F^i\left(c^T(\sigma_T), \mathcal{R}_T^i\left(V_{T+1}\left(c^T\right)\right)\right) \\ &= F^i\left(c^T(\sigma_T), \mathcal{R}_T^i\left(V_{T+1}(c')\right)\right) \\ &\geq F^i\left(c'(\sigma_T), \mathcal{R}_T^i\left(V_{T+1}(c')\right)\right) = V_{\sigma_T}^i(c'). \end{aligned} \tag{21}$$

Similarly, for all $\sigma_{T-1} \in \Sigma^{T-1}$, we have

$$\begin{aligned}
V^i_{\sigma_{T-1}}\left(c^T\right) &= F^i\left(c^T\left(\sigma_{T-1}\right), \mathcal{R}^i_{T-1}\left(V_T\left(c^T\right)\right)\right) \\
&\geq F^i\left(c'\left(\sigma_{T-1}\right), \mathcal{R}^i_{T-1}\left(V_T\left(c'\right)\right)\right) = V^i_{\sigma_{T-1}}\left(c'\right),
\end{aligned}$$

where the inequality comes from (21), the monotonicity of F in both arguments by Assumption 1 and because $\mathcal{R}^i$ is increasing in continuation utilities by Assumption 2. By induction, we have

$$V^i_{\sigma_0}\left(c^T\right) \geq V^i_{\sigma_0}\left(c'\right).$$

Notice that as $T \to +\infty$, we have $c^T \to c$ in the product topology.[27] $V^i_{\sigma_0}$ is continuous in product topology, so we have

$$V^i_{\sigma_0}\left(c\right) = \lim_{T \to +\infty} V^i_{\sigma_0}\left(c^T\right) \geq V^i_{\sigma_0}\left(c'\right),$$

which proves that $\succeq_i$ satisfies weak monotonicity, that is, $c \geq c'$ implies $c \succeq_i c'$. To prove the strong monotonicity, we note that for any $k > 0$, we have

$$\begin{aligned}
V^i_{\sigma_0}\left(c+k\right) &= F^i\left(c\left(\sigma_0\right)+k, \mathcal{R}^i_0\left(V_1\left(c+k\right)\right)\right) \\
&> F^i\left(c\left(\sigma_0\right), \mathcal{R}^i_0\left(V_1\left(c+k\right)\right)\right) \\
&\geq F^i\left(c\left(\sigma_0\right), \mathcal{R}^i_0\left(V_1\left(c\right)\right)\right) = V^i_{\sigma_0}\left(c\right),
\end{aligned}$$

where the strict inequality comes from $F^i_1 > 0$ and the weak inequality comes from the weak monotonicity of $\succeq_i$. □

Remark 2. Lemma 6 shows that V_0 is monotonic in c. From the proof, it is straightforward that V_t is monotonic in c for all $t \in \mathbb{T}$, that is, if $c \geq c'$ from time t on, then we have $V_t(c) \geq V_t(c')$.

Lemma 7. *Under Assumptions 1, 2, 4 and 5, $\succeq_i$ satisfies weak convexity.*

Proof. Suppose that $c \neq c'$, $c \succeq_i c'$, and $\alpha \in (0,1)$. Again, we define c^T as in (20), and the concavity of F implies that for all $\sigma_T \in \Sigma^T$, we have

$$\begin{aligned}
V^i_{\sigma_T}\left(\alpha c^T + (1-\alpha)c'\right) &= F^i\left(\alpha c^T\left(\sigma_T\right) + (1-\alpha)c'\left(\sigma_T\right), \mathcal{R}^i_T\left(V_{T+1}\left(c'\right)\right)\right) \\
&\geq \alpha F^i\left(\alpha c^T\left(\sigma_T\right), \mathcal{R}^i_T\left(V_{T+1}\left(c'\right)\right)\right) + (1-\alpha)F^i\left(c'\left(\sigma_T\right), \mathcal{R}^i_T\left(V_{T+1}\left(c'\right)\right)\right) \\
&= \alpha V^i_{\sigma_T}\left(c^T\right) + (1-\alpha)V^i_{\sigma_T}\left(c'\right) \qquad (22)
\end{aligned}$$

[27] That is, $\lim_{T\to+\infty} ||\pi_t\left(c^T\right) - \pi_t\left(c\right)||_\infty = 0$ for all $t \geq 1$, where $\pi_t\left(c\right)$ denotes the time-t projection of c.

For all $\sigma_{T-1} \in \Sigma^{T-1}$, we have

$$
\begin{aligned}
& V_{\sigma_{T-1}}^{i}\left(\alpha c^{T}+(1-\alpha) c'\right) \\
=& F^{i}\left(\alpha c^{T}\left(\sigma_{T-1}\right)+(1-\alpha) c'\left(\sigma_{T-1}\right), \mathcal{R}_{T-1}^{i}\left(V_{T}\left(\alpha c^{T}+(1-\alpha) c'\right)\right)\right) && (23) \\
\geq & F^{i}\left(\alpha c^{T}\left(\sigma_{T-1}\right)+(1-\alpha) c'\left(\sigma_{T-1}\right), \alpha \mathcal{R}_{T-1}^{i}\left(V_{\sigma_{T}}^{i}\left(c^{T}\right)\right)+(1-\alpha) \mathcal{R}_{T-1}^{i}\left(V_{\sigma_{T}}^{i}\left(c'\right)\right)\right) && (24) \\
\geq & \alpha F^{i}\left(c^{T}\left(\sigma_{T-1}\right), \mathcal{R}_{T-1}^{i}\left(V_{\sigma_{T}}^{i}\left(c^{T}\right)\right)\right)+(1-\alpha) F^{i}\left(c'\left(\sigma_{T-1}\right), \mathcal{R}_{T-1}^{i}\left(V_{\sigma_{T}}^{i}\left(c'\right)\right)\right) && (25) \\
=& \alpha V_{\sigma_{T-1}}^{i}\left(c^{T}\right)+(1-\alpha) V_{\sigma_{T-1}}^{i}\left(c'\right),
\end{aligned}
$$

where (24) follows from (22) and the monotonicity and concavity of $\mathcal{R}_{T-1}^{i}$, and (25) comes from the concavity of F^{i}. By induction, we have

$$
V_{\sigma_{0}}^{i}\left(\alpha c^{T}+(1-\alpha) c'\right) \geq \alpha V_{\sigma_{0}}^{i}\left(c^{T}\right)+(1-\alpha) V_{\sigma_{0}}^{i}\left(c'\right).
$$

Again, since $c^{T} \rightarrow c$ in product topology and V^{i} is continuous in the product topology, we must have

$$
V_{\sigma_{0}}^{i}\left(\alpha c+(1-\alpha) c'\right) \geq \alpha V_{\sigma_{0}}^{i}(c)+(1-\alpha) V_{\sigma_{0}}^{i}\left(c'\right),
$$

which proves that $\succeq$ is convex. $\square$

Proof of Theorem 1

Proof. Let $B=\Sigma \times \mathbb{T}$, $\mathcal{B}=\mathcal{F} \otimes 2^{\mathbb{T}}$ and $\upsilon=\mathbb{P}^{\pi_{0}} \otimes \lambda$, where λ is the counting measure on $\left(\mathbb{T}, 2^{\mathbb{T}}\right)$. Conditions (1) and (2) in Theorem 8 hold as $C_{i}=L_{\infty}^{+}(B, \mathcal{B}, \upsilon)$ for all $i \in I$. Condition (3) follows by our Assumption 3. Lemma 5 shows preferences are continuous in the product topology. Since the product topology is coarser than the Mackey topology, it follows preference are continuous in the Mackey topology as well and so (a) in condition (4) holds.[28]. Finally, Lemmas 6 and 7 show preferences are strongly monotone and weakly convex, respectively, and so (b) and (c) in condition (4) hold. $\square$

A.4 Proof of Lemma 1

Proof. The proof consists of the following four steps.

Step 1: We show that $\mathcal{R}_{t}^{i}\left(V_{t+1}^{i}\right)$ belongs to a compact set almost surely. Let $\bar{e} \equiv I \times M$ denote an upper bound of the aggregate endowment. Denote by $\bar{v}^{i} \equiv \operatorname{ess\,sup}_{\sigma, t}\left|V_{\sigma_{t}}^{i}(\bar{e})\right|$ and by $\underline{v}^{i} \equiv \operatorname{ess\,inf}_{\sigma, t}\left|V_{\sigma_{t}}^{i}(0)\right|$. From Lemma 6 (and Remark 2), we know that $V_{\sigma_{t}}^{i}(c)$ is monotonic in c, so $V_{\sigma_{t}}^{i}(c) \in\left[V_{\sigma_{t}}^{i}(0), V_{\sigma_{t}}^{i}(\bar{e})\right]$, and hence $V_{t}^{i} \in\left[\underline{v}^{i}, \bar{v}^{i}\right] \equiv \mathcal{V}^{i}$ $\mathbb{P}$-almost surely. As a consequence,

$$
\mathcal{R}_{t}^{i}\left(V_{t+1}^{i}\right) \in\left[\mathcal{R}_{t}^{i}\left(\underline{v}^{i}\right), \mathcal{R}_{t}^{i}\left(\bar{v}^{i}\right)\right]=\left[\underline{v}^{i}, \bar{v}^{i}\right]=\mathcal{V}^{i} \quad \mathbb{P}-a.s.,
$$

[28]See Lemma 2.48 in Aliprantis and Border (1999)

which comes from the monotonicity of $\mathcal{R}_t^i$ and the fact that $\mathcal{R}_t^i (k \cdot \mathbf{1}) = k$. By assumption, $V^i(c) \in L_\infty^+$ for all $c \in \mathbb{C}$, so $0 \le \underline{v}^i \le \overline{v}^i < \infty$, and hence $\mathcal{V}^i$ is compact.

Step 2: We show that $F_1^i \left(c^i(\sigma_t), \mathcal{R}_t^i \left(V_{t+1}^i\right)\right)$ is almost surely bounded away from 0. Let $\mathcal{C} \equiv [0, \overline{e}]$, which is the feasible set of consumption. From Assumption 1, $F_1^i : \mathcal{C} \times \mathcal{V}^i \to (0, +\infty]$ is continuous and hence lower semicontinuous. Using the generalization of the Weierstrass' theorem to lower semicontinuous function where $+\infty$ is allowed as a value (see Lemma 2.40 in Aliprantis and Border (1999)), we know that F_1^i obtains its minimum on $\mathcal{C} \times \mathcal{V}^i$, so there exists some $f_1 > 0$ such that $F_1^i \left(c^i(\sigma_t), \mathcal{R}_t^i \left(V_{t+1}^i\right)\right) \ge f_1$ almost surely.

Step 3: We have $L^{ij}(\sigma_t) \to +\infty$ almost surely implies $F_1^i \left(c^i(\sigma_t), \mathcal{R}_t^i \left(V_{t+1}^i\right)\right) \to +\infty$. This is a direct implication from Step 2.

Step 4: We show that $F_1^i \left(c^i(\sigma_t), \mathcal{R}_t^i \left(V_{t+1}^i\right)\right) \to +\infty$ almost surely implies $c^i(\sigma_t) \to 0$. Suppose not, then on a positive-probability set of paths, we can find an $\varepsilon > 0$ and an infinite sequence $\left\{c^i(\sigma_{t_k})\right\}_{k=1}^{\infty}$ such that $c^i(\sigma_{t_k}) \ge \varepsilon$ for each k. Let $\mathcal{C}_\varepsilon \equiv [\varepsilon, \overline{e}]$, and we have

$$F_1^i \left(c^i(\sigma_{t_k}), \mathcal{R}_{t_k}^i \left(V_{t_k+1}^i\right)\right) \le \max_{(c,\mathcal{R}) \in \mathcal{C}_\varepsilon \times \mathcal{V}^i} F_1^i (c, \mathcal{R}) < +\infty, \quad \mathbb{P} - a.s.,$$

where the maximum can be obtained since $F_1^i : \mathcal{C}_\varepsilon \times \mathcal{V}^i \to (0, +\infty)$ is a continuous function on a compact domain, and hence the standard Weierstrass' theorem applies. However, this fact contradicts $F_1^i \left(c^i(\sigma_t), \mathcal{R}_t^i \left(V_{t+1}^i\right)\right) \to +\infty$, so we must have $c^i(\sigma_t) \to 0$. $\square$

A.5 Discussion on Assumption 7

Below is a condition on exogenous variables that is sufficient for Assumption 7 to hold.

Proposition 3. *There exists some $M \in (0,1)$ such that Assumption 7 holds if for all $i \in I$ and $t \in \mathbb{T}$, we have*

$$\sup_{V \in \mathcal{V}} \left[\max_{j \ne i} \frac{\nabla_s \mathcal{R}_t^j (V)}{\nabla_s \mathcal{R}_t^i (V)} \right] \le M \quad \textit{for some } s \in S,$$

where $\mathcal{V}$ denotes a compact set containing all possible utility values.

Proposition 3 says that one sufficient condition for Assumption 7 to hold is that there exists some state $s \in S$ at which every consumer's effective belief is sufficiently different from other consumers.

Proof. By definition, we have

$$\begin{aligned} \max_{j \ne i} L_{t+1}^{ij} &\le \max_{j \ne i} L_t^{ij} \times \max_{j \ne i} \frac{\nabla_s \mathcal{R}_t^j (V_{t+1})}{\nabla_s \mathcal{R}_t^i (V_{t+1})} \times \max_{j \ne i} D^{ji}(\sigma_t) \\ &\le \max_{j \ne i} L_t^{ij} \times M \times d \quad \text{when } s_{t+1} = s, \end{aligned}$$

where $d < \infty$ denotes the upper bound of D^{ji}. Let $M < 1/d$. It is easy to see that when $\max_{j \neq i} L_t^{ij} < \overline{L}$, then after $K = \left\lceil \log \frac{\underline{L}/\overline{L}}{Md} \right\rceil$ realizations of state s, we have $\max_{j \neq i} L_{t+K}^{ij} < \underline{L}$, so Assumption 7 is satisfied. $\square$

To get a more concrete idea, we employ Proposition 3 to provide conditions for smooth ambiguity preferences below. Other preferences can be discussed analogously.

Example 19. (Smooth ambiguity preferences) Suppose that consumers have smooth ambiguity preferences with i.i.d. beliefs,

$$V_t(c) = u(c_t) + \beta \times \phi^{-1}\left(\mathbb{E}_\mu \phi\left[\mathbb{E}_\pi\left(V_{t+1}(c)\right)\right]\right).$$

Denote by $m = \min_i \frac{\phi_i'(\overline{V})}{\phi_i'(\underline{V})}$, $M = \max_i \frac{\phi_i'(\underline{V})}{\phi_i'(\overline{V})}$ and $d = \frac{\min_i \beta_i}{\max_i \beta_i}$, where $\underline{V}$ and $\overline{V}$ denote the lowest and the highest possible utility levels. If for all i, we have

$$\max_{j \neq i} \frac{\pi^j(s)}{\pi^i(s)} < \frac{m}{M} \times d \quad \text{for some } s \in S, \tag{26}$$

where $\pi^i(s) \equiv \sum_\pi \mu^i(\pi) \times \pi(s)$, then Assumption 7 is satisfied.

A.6 Auxiliary Results

Recall that consumption dynamics are determined by

$$L_{t+1}^{ij} = L_t^{ij} \times B^{ji}(\sigma_t, s_{t+1}) \times D^{ji}(\sigma_t).$$

To characterize the dynamics, it is more convenient to consider a modified process $\left\{\hat{L}_t^{ij}\right\}_{t=1}^{\infty}$ where

$$\hat{L}_{t+1}^{ij} = \hat{L}_t^{ij} \times B^{ji}(\sigma_t, s_{t+1}) \times D^{ji}(\sigma_t, s_{t+1}).$$

We first notice that these two processes are *comparable* in the sense that their ratios are uniformly bounded.

Lemma 8. *(Comparability) There exists* $0 < \alpha < \beta < +\infty$ *such that* $\alpha < L_t^{ij}/\hat{L}_t^{ij} < \beta$ *for all* $t \geq 1$.

Proof. By definition, we have

$$\frac{L^{ij}(\sigma_t)}{\hat{L}^{ij}(\sigma_t)} = \frac{\prod_{\tau=0}^{t-1} B^{ji}(\sigma_{\tau+1}) \times D^{ji}(\sigma_\tau)}{\prod_{\tau=0}^{t-1} B^{ji}(\sigma_{\tau+1}) \times D^{ji}(\sigma_{\tau+1})} = \frac{D^{ji}(\sigma_0)}{D^{ji}(\sigma_t)}.$$

Note that

$$D^{ji}(\sigma_t) = \frac{F_2^j\left(c(\sigma_t), \mathcal{R}_t^j\left(V_{t+1}^j\right)\right)}{F_2^i\left(c(\sigma_t), \mathcal{R}_t^i\left(V_{t+1}^i\right)\right)} \in (0, +\infty),$$

because consumption and utility are bounded, and $F_2 \in (0, +\infty)$ and is continuous. So, there exists $0 < \underline{d} < \overline{d} < +\infty$ that bound D^{ij} from below and above, which proves the lemma. $\square$

Lemma 9. *(Continuity) For all $i, j \in I$ and $t \in \mathbb{T}$, and $\varepsilon > 0$, there exists a uniform $\overline{L} \in \mathbb{R}_+$ such that*

$$\hat{L}_t^{ij} > \overline{L} \Longrightarrow |V_t^i - \underline{V}^i| < \varepsilon.$$

Remark 3. Lemma 9 says that when the (modified) marginal utility ratio between i and j becomes very large, the continuation utility of i at time t becomes very close to the utility of consuming zero forever.

Proof. First, note that V_t^i is continuous in product topology by Lemma 5, so for all $\varepsilon > 0$, there exists some $K \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\max_{\omega \in \Sigma_t} \max_{0 \le k \le K} c_{t+k}^i(\omega) < \epsilon \Longrightarrow |V_t^i - \underline{V}^i| < \varepsilon. \tag{27}$$

Second, we can show that fixing $\epsilon > 0$ and $K \in \mathbb{N}$, there exists some $L_0 \in \mathbb{R}_+$ such that

$$\min\left\{L_t^{ij}, ..., L_{t+K}^{ij}\right\} > L_0 \Longrightarrow \max\left\{c_t^i, ..., c_{t+K}^i\right\} < \epsilon. \tag{28}$$

This comes from the fact that for all $\tau \in \mathbb{T}$,

$$L_\tau^{ij} > L_0 \Rightarrow F_1^i\left(c_\tau^i, \mathcal{R}_\tau^i\right) > L_0 \times F_1^j\left(c_\tau^j, \mathcal{R}_\tau^j\right),$$

and

$$F_1^j\left(c_\tau^j, \mathcal{R}_\tau^j\right) > \min\left\{F_1^j\left(\overline{e}, V\right) : V \in \mathcal{V}^j\right\} \equiv \underline{F}_1^j > 0,$$

where $\mathcal{V}^j = \left[\underline{V}^j, \max_s \overline{V}_s^j\right]$ stands for the set of possible utility values for consumer j. Note that $\mathcal{R}_\tau^i \in \mathcal{V}^i$ has a compact range, and F^i satisfies the Inada condition, so we can choose L_0 sufficiently large such that

$$F_1^i\left(c_\tau^i, \mathcal{R}_\tau^i\right) > L_0 \times \underline{F}_1^j \Rightarrow c_\tau^i < \epsilon,$$

which proves (28). Lemma 8 then implies that there exists $\overline{L}$ such that

$$\min\left\{\hat{L}_t^{ij}, ..., \hat{L}_{t+K}^{ij}\right\} > \overline{L} \Longrightarrow \min\left\{L_t^{ij}, ..., L_{t+K}^{ij}\right\} > L_0, \tag{29}$$

so Lemma 9 is proved after combining (27), (28) and (29). $\square$

A.7 Proof of Proposition 2

A.7.1 Proof of Proposition 2 (i)

Lemma 10. *When $j \succ_i i$, there exists $\overline{L} < \infty$ such that for all $t \in \mathbb{T}$, we have*

$$\mathbb{P}\left(\hat{L}_\tau^{-i} \leq \overline{L}, \text{for some } \tau \geq t | \hat{L}_t^{-i} > \overline{L}\right) = 1, \tag{30}$$

where $\hat{L}^{-i}(\sigma_t) = \min_{k \neq i} \hat{L}^{ki}(\sigma_t)$.

Remark 4. $\hat{L}_t^{-i} \in \left(\overline{L}, +\infty\right)$ means that consumer i almost dominates the market, so Lemma 10 says that if consumer i is locally dominated by another consumer, then i can't dominate the market with positive probability.

Proof. Suppose (30) doesn't hold, then there exists $t \in \mathbb{T}$ and corresponding σ_t such that for all $\overline{L} < \infty$, we have

$$\mathbb{P}\left(\hat{L}_\tau^{-i} > \overline{L}, \text{for all } \tau \geq t | \hat{L}_t^{-i} > \overline{L}\right) > 0, \tag{31}$$

i.e., consumer i remains her dominant position in all future periods. From Lemma 9, we can choose $\overline{L}$ sufficiently large such that whenever $\hat{L}_t^{-i} > \overline{L}$, V_t^i is very close to $\overline{V}^i$, and V_t^k is very close to $\underline{V}^k$ for $k \neq i$. Let the local discount rate ratio and the local belief ratio be defined as

$$D_i^{ji}(s) = \frac{F_2^j\left(0, \mathcal{R}^j\left(\underline{V}^j\right)\right)}{F_2^i\left(e(s), \mathcal{R}^i\left(\overline{V}^i\right)\right)} \text{ and } B_i^{ji}(s) = \nabla_s \mathcal{R}^j\left(\underline{V}^j\right) / \nabla_s \mathcal{R}^i\left(\overline{V}^i\right), \tag{32}$$

which are the effective discount rate and belief ratios between j and i when consumer i consumes all the endowment forever. When $\hat{L}_\tau^{-i} > \overline{L}$ for all $\tau \geq t$, and when $\overline{L}$ is very large, the generalized discount rate ratio, $D^{ij}(\sigma_\tau)$, and belief ratio, $B^{ij}(\sigma_\tau)$, are very close to the local discount rate ratio $D_i^{ij}(s_\tau)$ and belief ratio $B_i^{ij}(s_\tau)$ for all $\tau \geq t$. Formally, for all $\varepsilon > 0$, there exists some $\overline{L} < \infty$ such that for all $\tau \geq t$, we have

$$\begin{aligned}
\log\left(\hat{L}_{\tau+1}^{ji}\right) &= \log\left(\hat{L}_\tau^{ji}\right) + \log\left[B^{ij}(\sigma_\tau, s_{\tau+1}) D^{ij}(\sigma_\tau, s_{\tau+1})\right] \\
&\leq \log\left(\hat{L}_\tau^{ji}\right) + (1+\varepsilon)\left[\log B_i^{ij}(s_{\tau+1}) D_j^{ij}(s_{\tau+1})\right].
\end{aligned}$$

Taking the limit of the time average, we have

$$\begin{aligned}
\lim_{T\to+\infty} \frac{1}{T-t} \sum_{k=t}^{T} \log\left(\hat{L}_{\tau+1}^{ji}\right) &\leq \lim_{T\to+\infty} \frac{1}{T-t} \log\left(\hat{L}_\tau^{ji}\right) + \lim_{T\to+\infty} \frac{1+\varepsilon}{T-t} \sum_{k=t}^{T} \left[\log B_i^{ji}(s_{\tau+1}) D_j^{ji}(s_{\tau+1})\right] \\
&= (1+\varepsilon)\,\mathbb{E}\left[\log D_i^{ij}(s_\tau) B_i^{ij}(s_\tau)\right] = (1+\varepsilon)\left(\mathcal{S}_i^i - \mathcal{S}_i^j\right) < 0 \quad \mathbb{P}-a.s.,
\end{aligned} \tag{33}$$

where the convergence comes from the strong law of large numbers, and the inequality comes from

$j \succ_i i$. However, (33) implies that $\hat{L}^{ji}_{\tau} \to 0$ almost surely, which contradicts (31), so we must have (30). $\square$

Proposition 2 (i) then follows directly from Lemma 10, because if i dominates the market with positive probability, we must have $\hat{L}^{-i}_t \to +\infty$ with positive probability, which contradicts Lemma 10.

A.7.2 Proof of Proposition 2 (ii)

For some $\underline{L} > 0$, we define $T = \inf\left\{\tau : \hat{L}^{i-}_{\tau} > \underline{L}\right\}$, where $\hat{L}^{i-}_{\tau} = \max_{j\neq i} \hat{L}^{ij}_{\tau}$. We construct a local martingale $m^{ij}_t = \left(\hat{L}^{ij}_{t\wedge T}\right)^{\rho}$. That is,

$$m^{ij}_{t+1} = \begin{cases} m^{ij}_t \times \left[B^{ji}\left(\sigma_t, s_{t+1}\right) \times D^{ji}\left(\sigma_t, s_{t+1}\right)\right]^{\rho} & t < T \\ m^{ij}_t & t \geq T \end{cases},$$

We present the following facts.

Fact 1. *If $i \succ_i j$, then there exists some $\rho > 0$ such that $\mathbb{E}\left[B^{ji}_i(s) \times D^{ji}_i(s)\right]^{\rho} < 1$.*

Proof. Let $X(s) = B^{ji}_i(s) \times D^{ji}_i(s)$, which is a bounded random variable. Then,

$$\lim_{\rho\to 0+} \mathbb{E}\left(\frac{X^{\rho}-1}{\rho}\right) = \mathbb{E}\left(\lim_{\rho\to 0+}\frac{X^{\rho}-1}{\rho}\right) = \mathbb{E}\left(\lim_{\rho\to 0+} X^{\rho} \times \log X\right) = \mathbb{E}\log X < 0,$$

where the first equality uses the dominated convergence theorem and the second one uses L'Hopital rule. Since $\mathbb{E}\left(\frac{X^{\rho}-1}{\rho}\right)$ is continuos in ρ, there exists $\rho > 0$ such that $\mathbb{E}\left(\frac{X^{\rho}-1}{\rho}\right) < 0$, that is $\mathbb{E}X^{\rho} < 1$. $\square$

Fact 2. *If $i \succ_i j$, there exists $\underline{L}, \rho > 0$ such that $\left\{m^{ij}_t\right\}$ is a supermartingale.*

Proof. Let ρ satisfy Fact 1. When $\hat{L}^{i-}_0 > \underline{L}$, the process is constant, and hence trivially a supermartingale. Suppose that $\hat{L}^{i-}_0 \leq \underline{L}$. From Lemma 9, when $\underline{L}$ is very small and when $\hat{L}^{i-}_t \leq \underline{L}$, both $B^{ji}(\sigma_t, s)$ and $D^{ji}(\sigma_t, s)$ are very close to $B^{ji}_i(s)$ and $D^{ji}_i(s)$. So for all $\varepsilon > 0$, there exists $\underline{L} > 0$ such that when $t < T$, we have

$$\mathbb{E}\left[m^{ij}_{t+1}|\mathcal{F}_t\right] \leq m^{ij}_t \times (1+\varepsilon)\,\mathbb{E}\left[B^{ji}_i(s) \times D^{ji}_i(s)\right]^{\rho}.$$

By Fact 1 we can choose $\varepsilon > 0$ to be sufficiently small so that $\mathbb{E}\left[m^{ij}_{t+1}|\mathcal{F}_t\right] \leq m^{ij}_t$ when $t < T$ $\left\{m^{ij}_t\right\}$ is a supermartingale. $\square$

Lemma 11. *If $i \succ_i j$ for all $j \neq i$, there exists $p > 0$ and $\varepsilon' > \varepsilon > 0$ such that for all $t \in \mathbb{T}$, we have*

$$\mathbb{P}\left(\hat{L}^{i-}_{\tau} \leq \varepsilon', \textit{for all } \tau \geq t | \hat{L}^{i-}_t \leq \varepsilon\right) > p. \tag{34}$$

Remark 5. Lemma 11 says that if i locally dominates everyone else, then conditional on i almost dominating the market, she will maintain the dominating position with a strictly positive probability. As showed later, this further implies that consumer i will (unconditionally) dominate the market with a strictly positive probability.

Proof. From Fact 2, $\left\{m_t^{ij}\right\}$ is a non-negative supermartingale, therefore it almost surely converges to some finite random variable m_∞^{ij} for all $j \neq i$. We have

$$\begin{aligned}\mathbb{P}\left(T<\infty\right) &\leq \mathbb{P}\left(\exists j\neq i: m_\infty^{ij} > \underline{L}^\rho\right) \leq \sum_{j\neq i}\mathbb{P}\left(m_\infty^{ij} > \underline{L}^\rho\right)\\ &\leq \sum_{j\neq i}\frac{\mathbb{E}m_\infty^{ij}}{\underline{L}^\rho} \leq (I-1)\times\left(\frac{\hat{L}_0^{i-}}{\underline{L}}\right)^\rho, \end{aligned} \tag{35}$$

where the third inequality comes from Markov inequality and the last one comes from the supermartingale property. Let $\epsilon \in (0,1)$, (35) implies that

$$\mathbb{P}\left(T=\infty|\hat{L}_0^{i-} \leq \left(\frac{\epsilon}{I-1}\right)^{1/\rho}\underline{L}\right) > 1-\epsilon > 0. \tag{36}$$

For all $t \in \mathbb{T}$ and $\hat{L}_t^{i-}$, we construct a stopped process $\left\{m_{t+k}^{ij}\right\}_{k=0}^{\infty}$ that starts from time t and applying the same arguments.[29] Then, we have

$$\mathbb{P}\left(\hat{L}_\tau^{i-} \leq \underline{L} \text{ for all } \tau \geq t|\hat{L}_t^{i-} \leq \left(\frac{\epsilon}{I-1}\right)^{1/\rho}\underline{L}\right) > 1-\epsilon, \tag{37}$$

so the lemma is proved by setting $\varepsilon = \left(\frac{\epsilon}{I-1}\right)^{1/\rho}\underline{L}$, $\varepsilon' = \underline{L}$ and $p = 1-\epsilon$. □

Next we show that Lemma 11 implies that consumer i will dominate the market with a strictly positive probability. First, Assumptions 7 implies that for all $\hat{L}_0^{i-} < \infty$, there exists some $t \in \mathbb{T}$ such that $\hat{L}_t^{i-} \leq \left(\frac{\epsilon}{I-1}\right)^{1/\rho}\underline{L}$ happens with a strictly positive probability. Therefore, (37) implies that there exists some $\delta > 0$ such that

$$\mathbb{P}\left(\hat{L}_\tau^{i-} \leq \underline{L} \text{ for all } \tau \geq t\right) > \delta. \tag{38}$$

The martingale convergence theorem and (38) imply that $\hat{L}_t^{ij}$ converges to some limit $\hat{L}_\infty^{ij} \in [0, \underline{L}]$ with a strictly positive probability for all $j \neq i$. Assumptions 6 and 7 imply that the limit can only be $\hat{L}_\infty^{ij} = 0$ (otherwise $\hat{L}_t^{ij}$ can always drift away from $\hat{L}_\infty^{ij}$ with a probability bounded away from 0, and hence unstable). So, (38) is equivalent to that consumer i dominates the market with a strictly positive probability.

[29] Formally, define $T = \inf\left\{\tau : \hat{L}_{\tau+t}^{i-} > \underline{L}\right\}$ and $m_{t+k}^{ij} = \left(\hat{L}_{t+\min\{k,T\}}^{ij}\right)^\rho$.

A.8 Proof of Theorem 2

Proof. (i) Suppose that $i \succ_i j$ and $i \succ_j j$. From $i \succ_j j$ and Lemma 10, we know that for all $t \in \mathbb{T}$,

$$\mathbb{P}\left(\hat{L}_\tau^{ij} \leq \overline{L} \text{ for some } \tau \geq t \middle| \hat{L}_t^{ij} > \overline{L}\right) = 1. \tag{39}$$

Assumption 7 imply that there exists $\delta > 0$ such that for all $t \in \mathbb{T}$,

$$\mathbb{P}\left(\hat{L}_{t+K}^{ij} \leq \left(\frac{\epsilon}{I-1}\right)^{1/\rho} \underline{L} \middle| \hat{L}_t^{ij} \leq \overline{L}\right) > \delta. \tag{40}$$

From (36), $i \succ_i j$ implies that for all $t \in \mathbb{T}$, we have

$$\mathbb{P}\left(\hat{L}_\tau^{ij} \leq \underline{L} \text{ for all } \tau \geq t \middle| \hat{L}_t^{ij} \leq \left(\frac{\epsilon}{I-1}\right)^{1/\rho} \underline{L}\right) > 1 - \epsilon > 0. \tag{41}$$

Denote by $E = \cup_{k=1}^{\infty} \left\{\hat{L}_t^{ij} \leq \underline{L} \text{ for all } t \geq k\right\}$. Combining (39), (40) and (41), we know that for all $t \in \mathbb{T}$, we have

$$\mathbb{P}\left(E|\mathcal{F}_t\right) \geq (1-\epsilon)\,\delta > 0,$$

which further implies that $\mathbb{P}(E) = 1$ by Levy's 0-1 law, i.e., $\hat{L}_t^{ij}$ will be trapped below $\underline{L}$ eventually with probability 1. We can choose $\underline{L}$ to be arbitrarily small, so we must have $\hat{L}_t^{ij} \to 0$ almost surely, which implies that $c_t^j \to 0$, i.e., consumer j vanishes and consumer i is the only survivor.

(ii) Suppose that $i \succ_j j$ and $j \succ_i i$. Proposition 2 implies that both i and j dominate with zero probability. That is, they survive simultaneously with probability 1.

(iii) Suppose that $i \succ_i j$ and $j \succ_j i$. Proposition 2 implies that both i and j dominate with a strictly positive probability. We define $E^i = \cup_{k=1}^{\infty} \left\{\hat{L}_t^{ij} \leq \underline{L} \text{ for all } t \geq k\right\}$ and $E^j = \cup_{k=1}^{\infty} \left\{\hat{L}_t^{ij} \geq 1/\underline{L} \text{ for all } t \geq k\right\}$. Following the proof in case (i), there exists some $p > 0$ such that for all $t \in \mathbb{T}$, we have

$$\mathbb{P}\left(E^i \cup E^j|\mathcal{F}_t\right) > p \Longrightarrow \mathbb{P}\left(E^i \cup E^j\right) = 1.$$

Similarly, because we can choose $\underline{L}$ to be arbitrarily small, it implies that

$$\mathbb{P}\left(\left\{\hat{L}_t^{ij} \to 0\right\} \cup \left\{\hat{L}_t^{ij} \to +\infty\right\}\right) = 1 \Rightarrow \mathbb{P}\left(\left\{c_t^j \to 0\right\} \cup \left\{c_t^i \to 0\right\}\right) = 1.$$

That is, with probability 1, one of the two consumers will dominate the market. Besides, both E^i and E^j occur with a strictly probability by Proposition 2, so both $\mathbb{P}\left(c_t^j \to 0\right)$ and $\mathbb{P}\left(c_t^i \to 0\right)$ occur with a strictly positive probability. □

A.9 Proof of Theorem 3

Lemma 12. *There exists some* $p > 0$ *and* $\varepsilon' > \varepsilon > 0$ *such that for all* i *with* $U_i \neq \emptyset$ *and all* $t \in \mathbb{T}$, *we have*

$$\mathbb{P}\left(\max_{j \in U_i} \hat{L}_T^{ij} \geq \varepsilon' \middle| \hat{L}_t^{i-} < \varepsilon \right) \geq p,$$

where $T = \inf \left\{ \tau > t : \hat{L}_\tau^{i-} \geq \varepsilon' \right\}$.

Remark 6. Lemma 12 improves Lemma 10. Recall that Lemma 10 says that if there is a consumer j such that $j \succ_i i$, then i can't dominate the market. Lemma 12 further says that when consumer i almost dominates the market, her dominating position will be destroyed by some other consumer j such that $j \succ_i i$ (i.e., the marginal utility consumption of i increases relative to some $j \in U_i$ and falls outside of the dominating neighborhood).

Proof. For all $t \in \mathbb{T}$ and given that $\hat{L}_t^{i-} < \varepsilon$, then we define a stopping time $T = \inf \left\{ \tau > t : \hat{L}_\tau^{i-} \geq \varepsilon' \right\}$, where $\varepsilon' > \varepsilon > 0$. For all i and j, define a stopped process $m_{t+k}^{ij} = \left(\hat{L}_{\min\{t+k,T\}}^{ij} \right)^\rho$. Fact 2 shows that there exists $\rho, \varepsilon' > 0$ such that $\left\{ m_{t+k}^{ij} \right\}$ is a supermartingale, and hence converges to some limit m_∞^{ij}. Doob's optional stopping theorem implies that

$$\varepsilon^\rho \geq \left(\hat{L}_t^{ij} \right)^\rho \geq \mathbb{E}\left(m_\infty^{ij} | \mathcal{F}_t \right) \geq (\varepsilon')^\rho \times \mathbb{P}_{\{T<\infty\}} \left(\hat{L}_T^{ij} \geq \varepsilon' | \mathcal{F}_t \right).$$

Therefore, we have

$$\mathbb{P}_{\{T<\infty\}} \left(\max_{j \in D_i} \hat{L}_T^{ij} \geq \varepsilon' \middle| \mathcal{F}_t \right) \leq \sum_{j \in D_i} \mathbb{P}_{\{T<\infty\}} \left(\hat{L}_T^{ij} \geq \varepsilon' \middle| \mathcal{F}_t \right) \leq I \left(\frac{\varepsilon}{\varepsilon'} \right)^\rho. \tag{42}$$

Suppose that $U_i \neq \emptyset$, then Lemma 10 implies that $\mathbb{P}(T = \infty) = 0$, that is,[30]

$$\mathbb{P}\left(\hat{L}_T^{i-} \geq \varepsilon' \middle| \mathcal{F}_t \right) = 1,$$

so (42) implies that

$$\begin{aligned} & \mathbb{P}\left(\max_{j \in D_i} \hat{L}_T^{ij} \geq \varepsilon' | \mathcal{F}_t \right) \leq I \left(\frac{\varepsilon}{\varepsilon'} \right)^\rho \\ \Longrightarrow & \mathbb{P}\left(\max_{j \in U_i} \hat{L}_T^{ij} \geq \varepsilon' \middle| \mathcal{F}_t \right) \geq 1 - I \left(\frac{\varepsilon}{\varepsilon'} \right)^\rho, \end{aligned} \tag{43}$$

Here, we choose ε sufficiently small relative to ε', so the probability on the R.H.S. of (43) is strictly positive, which implies the lemma. ☐

[30] Recall that $\hat{L}_\tau^{i-} = \frac{1}{\hat{L}_\tau^{-i}}$.

Lemma 13. *For all $\varepsilon > 0$, there is some $q > 0$ such that for all $i \in I$ with $U_i \neq \emptyset$ and all $t \in \mathbb{T}$, we have*

$$\mathbb{P}\left(\cup_{j \in U_i}\left\{\hat{L}_\tau^{j-} < \varepsilon \text{ for some } \tau \geq t\right\} | \hat{L}_t^{i-} < \varepsilon\right) \geq q. \tag{44}$$

Remark 7. Lemma 13 says that conditional on that i almost dominates the market, some other consumer who dominates i (if existing) will almost dominate the market as well sometime in the future.

Proof. From Lemma 12, there exists $p > 0$ such that

$$\mathbb{P}\left(\cup_{j \in U_i}\left\{\hat{L}_\tau^{ij} \geq \varepsilon' \text{ for some } \tau > t\right\} | L_t^{i-} < \varepsilon\right) \geq p. \tag{45}$$

Define $T = \inf\left\{\tau > t : \hat{L}_\tau^{i-} \geq \varepsilon'\right\}$. So we have: (i) $T < \infty$, and (ii) there exists some $j \in U_i$ such that $\hat{L}_T^{ij} \geq \varepsilon'$. Each such j must satisfy

$$\begin{aligned}
\hat{L}_T^{j-} &= \max_{k \neq j} \hat{L}_T^{jk} = \hat{L}_T^{ji} \times \max_{k \neq j} \hat{L}_T^{ik} \\
&\leq \frac{1}{\varepsilon'} \times \varepsilon' \max\left(D^{ki} B^{ki}\right) \leq \frac{1}{\varepsilon'} \times \varepsilon' M \text{ for some } M < \infty.
\end{aligned} \tag{46}$$

where the boundedness of D and B comes from the proof of Lemma 8. (45) and (46) imply that

$$\mathbb{P}\left(\cup_{j \in U_i}\left\{\hat{L}_\tau^{j-} \leq \varepsilon' M \text{ for some } \tau \geq t\right\} | L_t^{i-} < \varepsilon\right) \geq p.$$

Assumption 7 implies that conditional on $\hat{L}_t^{j-} \leq \varepsilon' M$, there exists $K < \infty$ signals after which we have $\hat{L}_{\tau+K}^{j-} < \varepsilon$. So, there exists some $q > 0$ such that probability of $\hat{L}_\tau^{j-} < \varepsilon$ for some $\tau > t$ is uniformly bounded by q. □

Next, we prove the following claim, which implies Theorem 3 directly.

Lemma 14. $\mathbb{P}\left(\cup_{i \in I^*}\left\{c_t^j \to 0 \text{ for all } j \neq i\right\}\right) = 1$, *with* $\mathbb{P}\left(c_t^j \to 0 \text{ for all } j \neq i\right) > 0$ *for all* $i \in I^*$.

Proof. Let $\hat{L} \equiv \left\{\hat{L}^{ij}, \text{ for all } i, j \in I\right\}$ and we define $B_\varepsilon^i = \left\{\hat{L} : \hat{L}^{i-} < \varepsilon\right\}$. From Assumption 7, we know that for all $t \in \mathbb{T}$, the probability that there exists some $j_1 \in I$ such that $\hat{L}_{t_1} \in B_\varepsilon^{j_1}$ for some $t_1 \geq t$ is uniformly greater than 0.[31] Then, when $\hat{L}_{t_1} \in B_\varepsilon^{j_1}$, and if $U_{j_1} \neq \emptyset$, Lemma 13 implies that with a strictly positive probability that exists some $t_2 > t_1$ such that $\hat{L}_{t_2} \in B_\varepsilon^{j_2}$ for some $j_2 \in U_{j_1}$. Applying the arguments iteratively, we know that with a strictly positive probability, $\hat{L}_t$ follows the trajectory $B_\varepsilon^{j_1} \to B_\varepsilon^{j_2} \to B_\varepsilon^{j_3} \to ....$, where $j_{m+1} \in U_{j_m}$ until it enters some $B_\varepsilon^{j_*}$ with $U_{j_*} = \emptyset$. Once $\hat{L}_t \in B_\varepsilon^{j_*}$, Lemma 11 implies that $\hat{L}_t$ is trapped inside $B_\varepsilon^{j_*}$ with a strictly positive probability, in which case consumer j^* dominates the market. Note that in each step, the probability is uniformly

[31] We simply choose $j_1 \in \arg\min_{j \in I} F_1^j\left(c_t^j, \mathcal{R}_t^j\right)$. By definition we have $L_t^{j_1-} = \max_i L_t^{ji} \leq 1 < \infty$. Assumption 7 implies that the probability that $L_{t_1}^{j_1-} < \varepsilon$ for some $t_1 > t$ is uniformly bounded from below.

bounded away from 0. Therefore, for all $t \in \mathbb{T}$ and $\mathcal{F}_t$, there exists some $p > 0$ and $i \in I^*$ such that

$$\mathbb{P}\left(\cup_{i\in I^*}\left\{c_t^j \to 0 \text{ for all } j \neq i\right\}|\mathcal{F}_t\right) > p,$$

i.e., the probability that some consumer in I^* eventually dominates the market is uniformly greater than 0. Levy's 0-1 law then implies that with probability 1, some consumer in I^* will eventually dominate. Besides, from Lemma 11 and Assumption 7, we know that each consumer in I^* dominates with a strictly positive probability. □

A.10 Proof of Theorem 4

Denote by E the event that all consumers in $I\backslash G$ vanish. Under the assumptions of Theorem 4, we have the following lemmas.

Lemma 15. *On E, $\mathbb{P}-$almost surely there exists some $i \in G$ such that $\hat{L}_t^{i-}$ doesn't converge.*

Proof. Suppose not, then it happens with a strictly positive probability that $\hat{L}_t^{i-}$ converges for all $i \in G$. We first notice that by Assumption 7, the limit $\hat{L}_\infty^{i-}$ can only be 0 or ∞ except for null events. We further note that on E, all consumers in $I\backslash G$ vanish, and since $U_i \neq \emptyset$ for all $i \in G$, no consumer in G dominate the market by Proposition 2, so $\hat{L}_\infty^{i-}$ cannot be zero. Therefore, $\hat{L}_\infty^{i-}$ must be ∞ for all $i \in G$. But then all consumers in G vanish as well, which is not possible by definition. □

Lemma 16. *Suppose that $\hat{L}_t^{i-}$ doesn't converge, then* $\liminf \hat{L}_t^{i-} = 0$ *except for $\mathbb{P}$-null events.*

Proof. Denote by G the event that $\hat{L}_t^{i-}$ doesn't converge, so we can rewrite it as

$$G = \cup_{m,n\in\mathbb{Z}_{++}}\left\{\hat{L}_t^{i-} \text{ crosses } \left[\frac{1}{m}, n\right] \text{ infinitely many times}\right\} \equiv \cup_{m,n\in\mathbb{Z}_{++}} G_{mn}.$$

Further denote by F the event that $\liminf \hat{L}_t^{i-} > 0$, which can be rewritten as

$$F = \cup_{k\in\mathbb{Z}++}\left\{\liminf \hat{L}_t^{i-} \geq 1/k\right\} \equiv \cup_{k\in\mathbb{Z}++} F_k.$$

Therefore, $G \cap F = \cup_{k,m,n\in\mathbb{Z}_{++}} G_{mn} \cap F_k$ denotes the event that $\hat{L}_t^{i-}$ doesn't converge and $\liminf \hat{L}_t^{i-} > 0$. Suppose that $k \leq m$, then by definition $G_{mn} \cap F_k$ is a null event. Suppose that $k > m$, then on G_{mn}, $\hat{L}_t^{i-}$ falls below $1/m$ infinitely often. By Assumption 7, whenever $\hat{L}_t^{i-} \leq 1/m$, then it will fall below $1/k$ with a strictly positive probability in the future. So, on G_{mn}, we have

$$\sum_t \mathbb{P}\left(\hat{L}_\tau^{i-} < 1/k \text{ for some } \tau > t|\mathcal{F}_t\right) = +\infty.$$

By Levy's extension of Borel-Cantelli Lemmas (See Chapter 12 of Williams (1991)), we almost surely have $\left\{\hat{L}_t^{i-} < 1/k \text{ i.o.}\right\}$ on G_{mn}, which implies $\mathbb{P}(G_{mn} \cap F_k) = 0$. The arguments hold for all

$k, m, n \in \mathbb{Z}_{++}$, so we further have $\mathbb{P}(G \cap F) = 0$, which implies that when $\hat{L}_t^{i-}$ doesn't converge, we have $\liminf \hat{L}_t^{i-} = 0$ except for null events. $\square$

Proof of Theorem 4

Proof. Combining Lemmas 15 and 16, we know that on E, there almost surely exists $i \in G$ such that $\hat{L}_t \in B_\varepsilon^i = \left\{\hat{L} \in \mathbb{R}_{++}^{I \times I} : \hat{L}^{i-} < \varepsilon\right\}$ happens infinitely often. Recall that Lemma 12 says that

$$\mathbb{P}\left(\max_{j \in U_i} \hat{L}_{T_t^i}^{ij} \geq \varepsilon' \middle| \hat{L}_t^{i-} < \varepsilon\right) \geq p,$$

where $T_t^i = \inf\left\{\tau > t : \hat{L}_\tau^{i-} \geq \varepsilon'\right\}$. Recall that T_t^i is $\mathbb{P}$-almost surely finite. Therefore on E, we almost surely have

$$\sum_t \mathbb{P}\left(\max_{j \in U_i} \hat{L}_{T_t^i}^{ij} \geq \varepsilon' \middle| \mathcal{F}_t\right) = +\infty$$

which, by Levy's extension of Borel-Cantelli Lemma, implies that $\max_{j \in U_i} \hat{L}_t^{ij} \geq \varepsilon'$ occurs infinitely often on E almost surely. By definition, all consumers in $I \backslash G$ vanish on E, so we have $\max_{j \in U_i \cap G} \hat{L}_t^{ij} \geq \varepsilon'$ occurs infinitely often on E.[32] For any $t \in \mathbb{T}$, suppose that $\hat{L}_t^{ij} \geq \varepsilon'$ for some $j \in G$, then $\hat{L}_t$ enters B_ε^j with a positive probability. Following the iterated arguments as in Lemma 14, $\hat{L}_t$ follows a trajectory $B_\varepsilon^{j_1} \to B_\varepsilon^{j_2} \to B_\varepsilon^{j_3} \to \ldots.$, where $j_{m+1} \in U_{j_m} \cap G$ for each m and finally enters B_ε^g for some $g \in G^*$ with a positive probability, and the probability is bounded away from 0 for all t. Therefore, on E, we have

$$\sum_t \mathbb{P}\left(\cup_{g \in G^*}\left\{\hat{L}_\tau \in B_\varepsilon^g \text{ for some } \tau > t\right\} | \mathcal{F}_t\right) = +\infty,$$

which implies that $\hat{L}_t \in B_\varepsilon^g$ for some $g \in G^*$ infinitely often almost surely. Lemma 12 says that conditional on $\hat{L}_t \in B_\varepsilon^g$, the probability that $\max_{j \in U_g} \hat{L}_{T_t^g}^{gj} \geq \varepsilon'$. Therefore, on E, we have

$$\sum_t \mathbb{P}\left(\cup_{g \in G^*}\left\{\max_{j \in U_g} \hat{L}_{T_t^g}^{gj} \geq \varepsilon'\right\} | \mathcal{F}_t\right) = +\infty, \tag{47}$$

which implies that $\hat{L}_{T_t^g}^{gj} \geq \varepsilon'$ for some $g \in G^*$and $j \in U_g$ infinitely often almost surely. Notice that by assumption, $U_g \neq \emptyset$ and $U_g \subset I/G$, so (47) implies $\hat{L}_{T_t^g}^{gj} \geq \varepsilon'$ infinitely often for some $g \in G^*$and $j \in I/G$, which further implies, once again, that some consumer $j \in I/G$ doesn't vanish on E almost surely, so E must be a null event. $\square$

[32]If there were some $j \in U_i \cap G^c$ such that $\hat{L}_{T_t^i}^{ij} \geq \varepsilon'$ occurs infinitely often, it would imply that j's consumption is greater than 0 infinitely often, which contradicts the definition of E. To see this, let $T = T_t^i$ and note that $\max_k \frac{F_{T-1}^i}{F_{T-1}^k} < \varepsilon$, which implies that $\max_k \frac{F_T^i}{F_T^k} < \varepsilon \times M$ for some $M < \infty$, so $F_T^i < \varepsilon M \cdot \min_k F_T^k$. Also note that $\frac{F_T^i}{F_T^j} > \varepsilon'$, so $F_T^j < F_T^i \cdot \frac{1}{\varepsilon'} < \frac{\varepsilon}{\varepsilon'} M \cdot \min_k F_T^k$, which implies that j's consumption must be greater than 0.

A.11 Proof of Lemma 2

We first state the formal definition of the distance below.

Definition 17. For economy $\mathcal{E}_S = (e, \pi_0, F, \mathcal{R})$ and $\mathcal{E}_{\hat{S}} = \left(\hat{e}, \hat{\pi}_0, \hat{F}, \hat{\mathcal{R}}\right)$ with $\hat{S} \supset S$. We define

$$||\mathcal{E}_S - \mathcal{E}_{\hat{S}}|| = \max\left\{d_e, d_\pi, d_F, d_{\mathcal{R}}\right\},$$

where

$$\begin{aligned}
d_e &= \max_{i\in I, s\in\hat{S}} |e_i(s) - \hat{e}_i(s)|, \text{ where } e_i(s) \equiv 0 \text{ for } s \in \hat{S}\backslash S \\
d_\pi &= \max_{s\in\hat{S}} |\pi_0(s) - \hat{\pi}_0(s)|, \text{ where } \pi_0(s) \equiv 0 \text{ for } s \in \hat{S}\backslash S \\
d_F &= \max_{c,y\in\mathcal{C}\times Y} |F(c,y) - \hat{F}(c,y)|, \\
d_{\mathcal{R}} &= \max_{i\in I, V\in\mathcal{V}} |\mathcal{R}^i(V) - \hat{\mathcal{R}}^i(V)|, \text{where } \mathcal{R}^i(V) \equiv \mathcal{R}^i\left(\{V_s\}_{s\in S}\right),
\end{aligned}$$

where $\mathcal{C} \equiv [0, \overline{e}]$ and $Y = [0, \overline{v}]$, where $\overline{e}$ is the maximum total endowment in these two economies and $\overline{v}$ denotes the maximum continuation utility, and $\mathcal{V} \equiv [0, \overline{\boldsymbol{v}}] \subset \mathbb{R}^{\hat{S}}$ where $\overline{\boldsymbol{v}} = \overline{v} \times 1 \in \mathbb{R}^{\hat{S}}$.

Next we prove Lemma 2

Proof. The proof is by construction. Let $\hat{S} = S \cup \{\hat{s}_1, ..., \hat{s}_I\}$. For some $\epsilon > 0$, we define

$$\hat{e}_i(s) = \begin{cases} e_i(s) & \text{when } s \in S \\ \epsilon & \text{when } s \in \hat{S}\backslash S \end{cases}, \text{ and } \hat{\pi}_0(s) = \begin{cases} (1-\epsilon)\pi_0(s) & \text{when } s \in S \\ \epsilon & \text{when } s \in \hat{S}\backslash S \end{cases}.$$

It is straightforward that $\hat{e}$ and $\hat{\pi}_0$ satisfy all relevant assumptions when ϵ is sufficiently small. We keep F unchanged, so $\hat{F}^i = F^i$. For any vector $V \subset \mathbb{R}_+^{\hat{S}}$, define

$$\hat{\mathcal{R}}^i(V) = \left(1 - \epsilon - \epsilon^2\right)\mathcal{R}^i(V|_S) + \epsilon V_{\hat{s}_i} + \sum_{j\neq i} \frac{\epsilon^2}{I-1} V_{\hat{s}_j},$$

where $V|_S \equiv (V_s)_{s\in S}$ is the restriction of V on S. By assumption, we have: (i) $\mathcal{R}^i$ is continuously differentiable and strictly increasing in V_s for $s \in S$, and (ii) $\mathcal{R}^i$ is concave and satisfies $\mathcal{R}^i(k) = k$. Therefore, $\hat{\mathcal{R}}^i$ also satisfies these properties on the extended domain. When ϵ is sufficiently small, the difference between $\hat{\mathcal{R}}^i$ and $\mathcal{R}^i$ is also sufficiently small. Last, we show that it also satisfies Assumption 7. Recall that consumption dynamics satisfy

$$L_{t+1}^{ij} = L_t^{ij} \times B^{ji}(\sigma_t, s_{t+1}) \times D^{ji}(\sigma_t).$$

For all $i \in I$, when $s_{t+1} = \hat{s}_i$, we have

$$B^{ji}\left(\sigma_t, s_{t+1}\right) = \frac{\nabla_{s_{t+1}} \mathcal{R}^j(V^i_{t+1})}{\nabla_{s_{t+1}} \mathcal{R}^i(V^i_{t+1})} = \frac{\epsilon}{I-1}, \quad \forall j \neq i.$$

Besides, D^{ji} is bounded by some constant $\overline{d}$ from the proof of Lemma 8. Therefore, when ϵ is sufficiently small, finitely many realizations of $\hat{s}_i$ make L_t^{i-} below any given threshold, so it satisfies Assumption 7. □

A.12 Proof of Theorem 5

Proof. (i) "if" part: Suppose that $i \succ_i r$, then we have

$$\mathbb{E} \log \left[\nabla_s \mathcal{R}^i \left(\overline{V}^i\right)\right] > \mathbb{E} \log \left(\pi_0\left(s\right)\right) \geq \mathbb{E} \log \left[\nabla_s \mathcal{R}^j \left(\underline{V}^j\right)\right] \quad \text{for all } \mathcal{R}^j \in \mathscr{R}, \tag{48}$$

where the first inequality comes from $i \succ_i r$, and the second inequality employs the facts that: (i) $\nabla \mathcal{R}^j \left(\underline{V}^j\right)$ is a probability distribution, which comes from Lemma 3 and that $\underline{V}^j$ is constant across states, and (ii) the entropy achieves its minimum at the true distribution π_0. (48) implies that $i \succ_i j$ for all $\mathcal{R}^j \in \mathscr{R}$. As a consequence, in all ε-perfect economies with small sufficiently small ε, we also have $i \succ_i j$ for all $\mathcal{R}^j \in \mathscr{R}$ by continuity arguments. The previous argument applies to all $j \neq i$, so we have $i \succ_i j$ for all $j \neq i$ for all $\left\{\mathcal{R}^j\right\}_{j \neq i} \in \mathscr{R}^{I-1}$. Therefore, consumer i dominates the market with a positive probability in ε-perfect economies with small ε for all $\left\{\mathcal{R}^j\right\}_{j \neq i} \in \mathscr{R}^I$, which establishes that $\mathcal{R}^i$ robustly survives.

(ii) "Only if" part: Suppose that $r \succ_i i$, we want to prove that $\mathcal{R}^i$ can't robustly survive. Let $\mathcal{R}^+ \in \mathscr{R}$ be any preference such that: (a) a consumer with $\mathcal{R}^+$ locally dominates r, and (b) the effective belief at certainty is correct, that is, $\nabla \mathcal{R}^+(k \cdot \mathbf{1}) = \pi_0$ for all $k \in \mathbb{R}_{++}$.[33] Consider the preference profile $\left\{\mathcal{R}^j\right\}_{j \neq i}$ where $\mathcal{R}^j = \mathcal{R}_+$ for all $j \neq i$. Under this construction, j's effective belief is correct at certainty, so when i dominates the market, we must have $j \succ_i i$ because $r \succ_i i$ by assumption. Therefore, the local upper contour set of i is $U_i = I \backslash \{i\}$. On the other hand, each j locally dominates r, so we have $j \succ_j r \succeq_j k$ for all $k \neq j$. Therefore, the local upper contour set of j is $U_j = \emptyset$ for all $j \neq i$. Theorem 3 implies that consumer i vanishes almost surely. As such, we find a preference profile $\left\{\mathcal{R}^j\right\}_{j \neq i}$ such that consumer i vanishes, so $\mathcal{R}^i$ can't robustly survive. □

A.13 Proof of Theorem 6

Case 1: $\mathbb{E} \log \left[\nabla_s \mathcal{R}^i \left(\overline{V}^i\right)\right] < \mathbb{E} \log \left[\nabla_s \mathcal{R}^i \left(\underline{V}^i\right)\right]$**.**

From the definition of D, we know that for all $\delta > 0$, there exists some $\epsilon > 0$ such that

$$D\left(\mathcal{R}^j, \mathcal{R}^i\right) < \epsilon \Rightarrow \left|\mathbb{E} \log \left[\frac{\nabla_s \mathcal{R}^i \left(\underline{V}^i\right)}{\nabla \mathcal{R}^j \left(\underline{V}^j\right)}\right]\right| < \delta, \tag{49}$$

[33]Such preference exists, e.g., HAAA utility with correct subjective belief as in Example 17

that is, the entropy of $\nabla\mathcal{R}^i\left(\underline{V}^i\right)$ and $\nabla\mathcal{R}^j\left(\underline{V}^j\right)$ can be arbitrarily close when $\mathcal{R}^i$ and $\mathcal{R}^j$ are sufficiently similar. Therefore, there exists $\epsilon > 0$ such that whenever $D\left(\mathcal{R}^j, \mathcal{R}^i\right) < \epsilon$, we have

$$\mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)\right] < \mathbb{E}\log\left[\nabla_s\mathcal{R}^j\left(\underline{V}^j\right)\right], \tag{50}$$

which implies that $j \succ_i i$, so consumer i dominates the market with 0 probability from Proposition 2, and hence $\mathcal{R}^i$ can't locally robustly dominate.

Case 2: $\mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)\right] > \mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)\right]$**.**

Similarly, for all $\delta > 0$, there exists some $\epsilon > 0$ such that

$$D\left(\mathcal{R}^j, \mathcal{R}^i\right) < \epsilon \Rightarrow \left|\mathbb{E}\log\left[\frac{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}{\nabla_s\mathcal{R}^j\left(\underline{V}^j\right)}\right]\right| < \delta \text{ and } \left|\mathbb{E}\log\left[\frac{\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^j\left(\overline{V}^j\right)}\right]\right| < \delta. \tag{51}$$

So, there exists $\epsilon > 0$ such that for all $\mathcal{R}^j, \mathcal{R}^k \in B_\epsilon\left(\mathcal{R}^i\right) \equiv \left\{\mathcal{R} \in \mathscr{R} : D\left(\mathcal{R}, \mathcal{R}^i\right) < \epsilon\right\}$, we have

$$\mathbb{E}\log\left[\nabla_s\mathcal{R}^j\left(\overline{V}^j\right)\right] > \mathbb{E}\log\left[\nabla_s\mathcal{R}^k\left(\underline{V}^k\right)\right]. \tag{52}$$

That is, for all $\mathcal{R}^j \in B_\epsilon\left(\mathcal{R}^i\right)$, we have $j \succ_j k$ for all $\mathcal{R}^k \in B_\epsilon\left(\mathcal{R}^i\right)$, so j dominates the market with a strictly positive probability, which implies that consumer i can't dominate the market with probability 1 from Proposition 2, and hence $\mathcal{R}^i$ can't locally robustly dominate.

Case 3: $\mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)\right] = \mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)\right]$**.**

To prove this case, we construct the following preference

$$\mathcal{R}^j(V) = \left(1 - \varepsilon_j\right)\mathcal{R}^i(V) + \varepsilon_j \times \mathcal{R}^*(V), \tag{53}$$

where $\varepsilon_j \in [0, 1]$ and $\mathcal{R}^*$ is a preference that satisfies $\mathbb{E}\log\left[\frac{\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right] > 0$, e.g., the HAAA preference with $\gamma \in (0, 1)$ (see Guerdjikova and Sciubba (2015)). Next, we want to prove that $j \succ_j i$ in some ε-perfected economy where ε can be arbitrarily small. We first state the following lemma.

Lemma 17. *For all $\mathcal{R}^i \in \mathscr{R}$, we have $\mathbb{E}\log\left[\frac{\nabla_s\mathcal{R}^j\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right] > 0$ for all $\varepsilon_j > 0$.*

Proof. Notice that

$$\begin{aligned}\mathbb{E}\log\left[\frac{\nabla_s\mathcal{R}^j\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right] &= \mathbb{E}\log\left[\frac{(1-\varepsilon_j)\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)+\varepsilon_j\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right]\\ &\geq (1-\varepsilon_j)\times\mathbb{E}\log\left(\frac{\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right)+\varepsilon_j\mathbb{E}\log\left(\frac{\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right)\\ &= \varepsilon_j\mathbb{E}\log\left(\frac{\nabla_s\mathcal{R}^*\left(\overline{V}^i\right)}{\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)}\right)>0. \qquad (54)\end{aligned}$$

□

Lemma 17 doesn't directly imply $j \succ_j i$ because we only have $\nabla_s\mathcal{R}^j\left(\overline{V}^i\right)$ instead of $\nabla_s\mathcal{R}^j\left(\overline{V}^j\right)$, and it is possible that $\overline{V}^j \neq \overline{V}^i$. However, it turns out that there always exists some ε-perfection in which $\overline{V}^j = \overline{V}^i$, which is stated as a lemma below. Denote by $\mathscr{U}$ the set of feasible utility functions, i.e., utility functions u such that satisfy Assumptions 1 and 4 are satisfied. We have the following lemma.

Lemma 18. *There exists $\overline{\varepsilon}$ such that if $\varepsilon_j \in (0,\overline{\varepsilon})$, then for all $\delta > 0$, there exists some $u_j \in \mathscr{U}$ with $||u_j - u_i|| < \delta$ satisfies $\overline{V}^j = \overline{V}^i$, where $||\cdot||$ denotes the sup-norm over some compact set $\mathcal{C}$.*

Proof. Construct the following utility function

$$u_j(x) \equiv u_i(x) + \beta\left[\mathcal{R}^i\left(\overline{V}^i\right) - \mathcal{R}^j\left(\overline{V}^i\right)\right].$$

Notice that $C_j \equiv \mathcal{R}\left(\overline{V}^i\right) - \mathcal{R}^j\left(\overline{V}^i\right) \to 0$ as $\varepsilon_j \to 0$. So, when ε_j is sufficiently small, we have $||u_j - u_i|| < \delta$. It is easy to verify that $u_j \in \mathscr{U}$ when ε_j is sufficiently small (because u_j is equal to some constant plus $u_i \in \mathscr{U}$, and when ε_j is sufficiently small, C_j is also small, so we can guarantee that $u_j(0) > 0$ as well). Notice that for all $s \in S$, we have

$$\begin{aligned}\overline{V}^i_s = u_i(e(s)) + \beta\times\mathcal{R}^i\left(\overline{V}^i\right) &= u_j(e(s)) - \beta\left[\mathcal{R}^i\left(\overline{V}^i\right) - \mathcal{R}^j\left(\overline{V}^i\right)\right] + \beta\times\mathcal{R}^i\left(\overline{V}^i\right)\\ &= u_j(e(s)) + \beta\mathcal{R}^j\left(\overline{V}^i\right),\end{aligned}$$

so $\overline{V}^i$ solves $V = F\left(e,\mathcal{R}^j(V)\right)$. By definition, we also have $\overline{V}^j = F\left(e,\mathcal{R}^j\left(\overline{V}^j\right)\right)$. Under our assumptions, $F\left(c,\mathcal{R}^j(V)\right)$ has a unique fixed point (see Marinacci and Montrucchio (2010)), so $\overline{V}^j = \overline{V}^i$. □

Combining Lemmas 17 and 18, we can construct some preference $\mathcal{R}^j$ such that $j \succ_j i$ in some ε-perfected economy where ε can be arbitrarily small. Suppose that we choose other consumers' preferences sufficiently close to i, we will have $j \succ_j k$ for all $k \neq j$, so consumer j dominates the market with a strictly positive probability, which establishes that $\mathcal{R}^i$ can't locally robustly dominate.

A.14 Proof of Theorem 7

The proof follows directly from Theorem 6. We consider the following two cases.

Case 1: $\mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)\right] < \mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)\right]$**.**

From (51), there exists $\epsilon > 0$ such that for all $\mathcal{R}^j, \mathcal{R}^k \in B_\epsilon\left(\mathcal{R}^i\right) \equiv \left\{\mathcal{R} \in \mathscr{R} : D\left(\mathcal{R}, \mathcal{R}^i\right) < \epsilon\right\}$, we have

$$\mathbb{E}\log\left[\nabla_s\mathcal{R}^j\left(\overline{V}^j\right)\right] < \mathbb{E}\log\left[\nabla_s\mathcal{R}^k\left(\underline{V}^k\right)\right]. \tag{55}$$

That is, for all $\mathcal{R}^j \in B_\epsilon\left(\mathcal{R}^i\right)$, we have $k \succ_j j$ for all $\mathcal{R}^k \in B_\epsilon\left(\mathcal{R}^i\right)$, so every consumer j is locally dominated by all other consumers, i.e., as in Figure 6 (b). Suppose $|I| = 2$, then both consumers co-exist almost surely from Theorem 2, so $\mathcal{R}^i$ locally robustly survives.

Case 2: $\mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\overline{V}^i\right)\right] > \mathbb{E}\log\left[\nabla_s\mathcal{R}^i\left(\underline{V}^i\right)\right]$**.**

From the case 2 in the proof of Theorem 6, every consumer j locally dominates by all other consumers, i.e., as in Figure 6 (a), so consumer i survives with a positive probability, and hence $\mathcal{R}^i$ locally robustly survives.